Original Paper

Three-dimensional analytical model for the straight-through flow in an all-electric glass melting furnace

Gerhard Hilbig
Hochschule für Architektur und Bauwesen, Sektion Baustoffverfahrenstechnik, WB Physik, Weimar (GDR)

This paper presents a three-dimensional analytical model for calculating the velocity field of an all-electric glass melting furnace which is caused by the mass current at the mean temperature $\bar{T}$.

Dreidimensionales analytisches Modell der Durchlaufströmung in einer vollelektrischen Glasschmelzwanne

Es wird ein dreidimensionales, analytisches Modell zur Berechnung des Geschwindigkeitsfeldes in einer vollelektrischen Glasschmelzwanne dargestellt, das durch den Massenstrom bei einer mittleren Temperatur $\bar{T}$ verursacht wird.

1. Introduction

Mathematical modelling of glass melting furnaces is a very complex problem. It comprises the calculations of the fields of temperature, velocity, concentration and other parameters in the furnace. In the last twenty years a great number of publications appeared in this field. Most of them dealt with two-dimensional fields. Only a few papers tried to calculate three-dimensional fields, the thesis of Ungan [1] being one of them. Numerical methods were used to solve the problem in all the published literature. Using primitive variables $\underline{v}$ and p the main problem is the calculation of the pressure for which no boundary conditions can be given. This is the reason why many authors use the MAC (Marker And Cell) method by Harlow and Welch [2]. Other authors use the stream function and vorticity. In these cases the authors can not formulate boundary conditions for these fields.

The view that the nonlinearity of the equations makes it impossible to solve the problems with analytical methods does not agree with the author's opinion. He believes, however, that under particular conditions it is possible to solve such problems by analytical methods. This is the case especially when the heat conduction of the melt is high enough so that the temperature field deviates only a little from the mean temperature. Experiments and numerical results show on the other side that the temperature fluctuations in such furnaces are very small. It must be noted that the projection-iteration method used by Riederer [3] and Werner [4] needs a homogenization function which could only be found for an all-electrically heated furnace. An expansion to fuel-heated furnaces was impossible at this time. This paper presents a three-dimensional analytical model for calculating the velocity field which is caused by the mass current at the mean temperature $\bar{T}$. Apparently, analytical methods have not yet been employed to describe the velocity fields in glass melting furnaces. Grossmann [5] especially encouraged the author to present this paper. In further papers the calculation of velocity fields caused by temperature distributions $\tilde{T}(x,y,z)$ will be presented. Another matter of concern of principal meaning is to show that no boundary conditions for the pressure are needed when solving the problem in this manner.

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2. All-electric glass melting furnace

Figure 1 shows the glass melting furnace. The areas of the side walls are denoted by A_1, A_2, A_3 and A_4, and the areas for the bottom and the top by A_5 and A_6. The geometry is determined by the quantities $2\,L$, $2\,B$ and $2\,H$. The velocity field in the furnace caused by the mass current $\dot{M}$ is described by $\underline{v}(x,y,z)$. The melting power characterized by the mass current $\dot{M}$ is defined by

$$\dot{M} = \varrho_B \int_0^{2L} \int_0^{2B} v_3(x,y,2\,H)\,\mathrm{d}x\,\mathrm{d}y = \varrho_B\, 2\,L\, 2\,B\, \bar{v}_{3\perp A_6}. \quad (1)$$

In stationary cases $\dot{M}$ can also be expressed by the relation described in equation (2)

$$\dot{M} = \varrho_B \int_0^{2B} \int_0^{2H} v_1(2\,L,y,z)\,\mathrm{d}y\,\mathrm{d}z = \varrho_B\, A^*\, \bar{v}_{1\perp A^*}. \quad (2)$$

$v_3(x,y,2\,H)$ and $v_1(2\,L,y,z)$ in the equations (1 and 2) are the components of the velocity field $\underline{v}$ in the

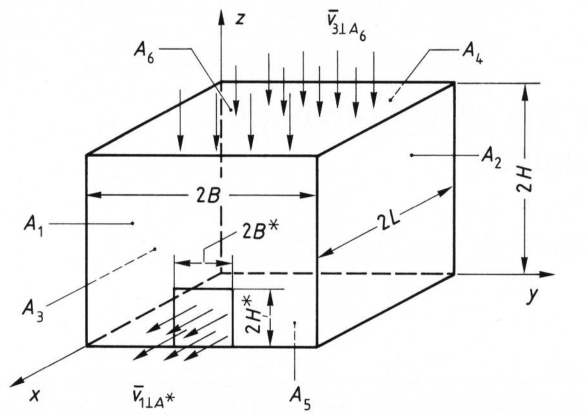

Figure 1. Melt region of the furnace. $2\,L$ = length, $2\,B$ = width, $2\,H$ = height, $2\,L^*$, $2\,B^*$, $2\,H^*$ = length, width and height in the throat area; A_i = boundaries; x,y,z = coordinates.

planes $z = 2\,H$ and $x = 2\,L$. $\bar{v}_{3\perp A6}$ and $\bar{v}_{1\perp A^*}$ are the mean velocity components in the areas A_6 and A^*. These can be used if the shape of the velocity distributions in these areas does not considerably influence the velocity field in the furnace, thus the following equations are obtained:

$$\underline{v}_{\perp A6} = -\,\bar{v}_{3\perp A6}\,\underline{e}_3 = -\frac{\dot{M}}{\varrho_B\,2\,L\,2\,B}\,\underline{e}_3\,, \tag{3}$$

$$\left.\begin{aligned} &\underline{v}_{\perp A3} = \bar{v}_{1\perp A^*}\,\underline{e}_1 = \frac{\dot{M}}{\varrho_B\,A^*}\,\underline{e}_1 && \text{in the throat,}\\ &\underline{v}_{\perp A3} = 0 && \text{without the throat.}\end{aligned}\right\} \tag{4}$$

3. Equation of motion

The basic equations for calculating the velocity field in fluids are the Navier-Stokes equations which for low Rayleigh numbers and the Boussinesq approximation have the following form

$$0 = -\,\nabla p + \varrho\,\underline{g} + \eta\,\triangle\,\underline{v}\,, \tag{5}$$

$$0 = \nabla\,\underline{v}\,. \tag{6}$$

In these equations $p(x,y,z)$ denote the pressure field, $\varrho = \varrho_B\,[1 - \beta\,(T - T_B)]$ the density, $T(x,y,z)$ the temperature field, β the volume expansion coefficient, T_B a reference temperature, $\underline{g} = -\,g\,\underline{e}_3$, $g = 9.81$ m/s^2, η the dynamic viscoscity, $\underline{v}(x,y,z)$ the velocity field, $\triangle$ the Laplace operator and ∇ the Nabla operator.

From equation (5) with

$$T(x,y,z) = \bar{T} + \tilde{T}(x,y,z)\,, \tag{7}$$

$$0 = -\nabla p + \varrho_B\,\underline{g} - \varrho_B\,\beta\,\underline{g}\,(\bar{T} - T_B + \tilde{T}) + \eta\,\triangle\,\underline{v} \tag{8}$$

is obtained where $\bar{T}$ represents the mean temperature of the melt given by

$$\bar{T} = \frac{1}{8\,L\,B\,H}\int_0^{2L}\int_0^{2B}\int_0^{2H} T(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{9}$$

while T_B is an arbitrarily eligible reference temperature. When $T = T_B$[1]) the following equation

$$0 = -\,\nabla\,p + \varrho_B\,\underline{g} - \varrho_B\,\beta\,\underline{g}\,\tilde{T} + \eta\,\triangle\,\underline{v} \tag{10}$$

is obtained. $\tilde{T}(x,y,z)$ in this equation denotes the temperature deviation from the mean temperature $\bar{T}$ of the melt. If these deviations are very small equation (11) holds

$$0 = -\,\nabla\,p' + \varrho_B\,\underline{g} + \eta\,\triangle\,\underline{v}'\,. \tag{11}$$

This equation yields in connection with the continuity equation $\nabla\,\underline{v}' = 0$, the velocity field $\underline{v}'(x,y,z)$ and the pressure field $p'(x,y,z)$ within the melt with the mean temperature $\bar{T}$. $\underline{v}'(x,y,z)$ in this case is called the passage flow or straight-through flow.

4. Pressure field

Using the divergence of the equation (11) the Laplace equation $\triangle\,p' = 0$ for the pressure field $p'(x,y,z)$ is yielded. The calculation of this field is only possible if boundary conditions for p' are given. Such conditions can not be formulated because the pressure distribution on the boundaries are caused by velocity $\underline{v}'$. This is the well-known problem of the pressure. The pressure distributions on the boundaries exist in reality, but it is impossible to specify them. By $p'_{\perp A_i}$ the really existing pressure distribution on the boundary A_i is denoted. In this case a well-defined mathematical problem is given which is to be solved by known analytical methods. The solution for a two-dimensional problem is given by Kneschke [6], Oezisik [7] and others. It is easy to expand this problem to three dimensions. The solution is given in section 8.1.

5. Velocity field

Equation (11) with the solution $p'(x,y,z)$ can now be written in the form

$$\triangle\,\underline{v}' = -\,\underline{n} \tag{12}$$

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[1]) $\bar{T}$ is related to the electric heating power by

$$N = (\bar{T} - T_A)\sum_l A_l\,R_l^{-1} + \dot{M}\,i$$

with heat resistance R_l of the wall with the area A_l and the specific melting enthalpy i. T_A denotes the atmospheric temperature.

where $\underline{n} = \eta^{-1}(\nabla p' - \varrho_B\, \underline{g})$. Equation (12) is a Poisson equation for the velocity field $\underline{v}'(x,y,z)$. $\underline{v}'$ now is the sum of the fields $\underline{u}(x,y,z)$ and $\underline{w}(x,y,z)$ where $\underline{u}$ and $\underline{w}$ are determined by the following equations

$$\triangle\, \underline{u} = 0 \tag{13}$$

with $\underline{u}_{\perp A_i} = \underline{v}'_{\perp A_i}$ and

$$\triangle\, \underline{w} = -\,\underline{n} \tag{14}$$

with $\underline{w}_{\perp A_i} = 0$ for all values of A_i.

The vector $\underline{u} = u_1\, \underline{e}_1 + u_2\, \underline{e}_2 + u_3\, \underline{e}_3$ has three components, thus three equations $\triangle\, u_1 = 0$, $\triangle\, u_2 = 0$ and $\triangle\, u_3 = 0$ have to be solved with the boundary conditions $u_{3\perp A_6} = -\,\overline{v}_{3\perp A_6}$ and $u_{3\perp A_i} = 0$ for A_1, A_2, A_3, A_4, A_5, $u_{1\perp A_3} = \overline{v}_{1\perp A_3}$ and $u_{1\perp A_i} = 0$ for $i = 1, 2, 4, 5, 6$. The boundary conditions for the component u_2 are $u_{2\perp A_i} = 0$ for $i = 1, 2, 3, 4, 5, 6$, thus the field $u_2(x,y,z)$ vanishes but not $v_2(x,y,z)$. The solutions for $u_1(x,y,z)$ and $u_3(x,y,z)$ are given in section 8.2.

The vector field $\underline{w} = w_1\, \underline{e}_1 + w_2\, \underline{e}_2 + w_3\, \underline{e}_3$ has also three components. In distinction to the field $\underline{u}$ the components w_1, w_2, w_3 must vanish on all boundaries A_i.

The field $w_1(x,y,z)$ follows from the equation $\triangle\, w_1 = -\, n_1$. Functions which are solutions of the Poisson equation and vanish on the boundaries of the furnace are sinus functions, thus equation (15)

$$w_1 = \sum_{l,m,n} a_{l,m,n}\, h_{l,m,n} \tag{15}$$

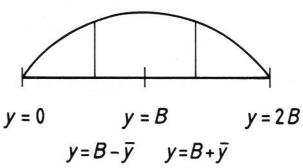

Figure 2. Symmetry relation.

is used with

$$h_{l,m,n} = h_l(x)\, h_m(y)\, h_n(z)\,, \tag{16}$$

$$h_l(x) = \sin\left(l\, \frac{\pi}{2\,L}\, x\right), \tag{17}$$

$$h_m(y) = \sin\left(m\, \frac{\pi}{2\,B}\, y\right), \tag{18}$$

$$h_n(z) = \sin\left(n\, \frac{\pi}{2\,H}\, z\right). \tag{19}$$

Introducing w_1 in $\triangle\, w_1 = -\, n_1$

$$\sum_{l,m,n} a_{l,m,n}\, N_{l,m,n}\, h_{l,m,n} = n_1 \tag{20}$$

is obtained with

$$N_{l,m,n} = \left(l\, \frac{\pi}{2\,L}\right)^2 + \left(m\, \frac{\pi}{2\,B}\right)^2 + \left(n\, \frac{\pi}{2\,H}\right)^2. \tag{21}$$

Multiplying equation (20) with $h_{\lambda,\mu,\nu}$ and integrating about x,y,z in the region $0 \le x \le 2\,L$, $0 \le y \le 2\,B$ and $0 \le z \le 2\,H$ yields

$$a_{\lambda,\mu,\nu}\, N_{\lambda,\mu,\nu}\, L\, B\, H = \int_0^{2L}\int_0^{2B}\int_0^{2H} n_1\, h_{\lambda,\mu,\nu}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = (n_1; h_{\lambda,\mu,\nu})^{2)} \tag{22}$$

if the orthogonality relations for the trigonometric functions are used. Equation (22) determines the coefficients $a_{\lambda,\mu,\nu}$, thus the field $w_1(x,y,z)$ is "known". The field $v_1'(x,y,z)$, however, must satisfy the symmetry condition shown in figure 2

$$v_1'(x, y = B + \overline{y}, z) = v_1'(x, y = B - \overline{y}, z) \tag{23}$$

which means that the integer μ has to be odd ($\mu = 1, 3, 5 \ldots$).

Analogous procedures yield the fields

$$w_2 = \sum_{o,p,q} b_{o,p,q}\, h_{o,p,q} \tag{24}$$

with

$$b_{o,p,q}\, N_{o,p,q}\, L\, B\, H = (n_2; h_{o,p,q})^{3)} \tag{25}$$

and

$$w_3 = \sum_{r,s,t} c_{r,s,t}\, h_{r,s,t} \tag{26}$$

with

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2) $(n_1; h_{\lambda,\mu,\nu}) = \left(\frac{1}{\eta}\, p_x'; h_{\lambda,\mu,\nu}\right) = \frac{1}{\eta}\int_0^{2L} \frac{\mathrm{d}}{\mathrm{d}x} F_{\mu,\nu}(x) \cdot h_\lambda(x)\, \mathrm{d}x\,.$

3) $(n_2; h_{\lambda,\mu,\nu}) = \left(\frac{1}{\eta}\, p_y'; h_{\lambda,\mu,\nu}\right) = \frac{1}{\eta}\int_0^{2B} \frac{\mathrm{d}}{\mathrm{d}y} G_{\lambda,\nu}(y) \cdot h_\mu(y)\, \mathrm{d}y\,.$

$$c_{r,s,t} N_{r,s,t} L B H = (n_3; h_{r,s,t})^{4)} . \tag{27}$$

The field $v_3'(x,y,z)$ also has to satisfy the condition

$$v_3'(x,y = B + \bar{y},z) = v_3'(x,y = B - \bar{y},z) \tag{28}$$

which only is possible if the integer s is odd ($s = 1, 3, 5, \ldots$). In distinction to v_1' and v_3' the field $v_2'(x,y,z) = w_2(x,y,z)$ has to meet the condition

$$w_2(x,y = B + \bar{y},z) = - w_2(x,y = B - \bar{y},z) \tag{29}$$

which is possible if the integer p is even ($p = 2, 4, 6, \ldots$).

The velocity field $\underline{v}' = \underline{u} + \underline{w}$ now still has to satisfy the continuity equation $\nabla \underline{v}' = 0$, $\nabla \underline{u} + \nabla \underline{w} = 0$ or

$$u_{1,x} + u_{3,z} + w_{1,x} + w_{2,y} + w_{3,z} = 0 . \tag{30}$$

Using the equations (15, 24 and 26) and u_1, u_3 from section 8.2., the equation (30) yields

$$\sum_{l,m,n} a_{l,m,n} h_l'(x) h_m(y) h_n(z) + \sum_{o,p,q} b_{o,p,q} h_o(x) h_p'(y) h_q(z) + \sum_{r,s,t} c_{r,s,t} h_r(x) h_s(y) h_t'(z) + \\ + \sum_{u,v} f_{u,v}'(x) h_u(y) h_v(z) + \sum_{i,j} k_{i,j}'(z) h_i(x) h_j(y) = 0^{5)} , \tag{31}$$

where m, s, u, j are odd integers and p is even. The generalized product from equation (31) with $h_\lambda'(x) h_\mu(y) h_\nu(z)$ yields

$$a_{\lambda,\mu,\nu} \left(\lambda \frac{\pi}{2L}\right)^2 L B H + \sum_{o,p} b_{o,p,\nu} I_{o,\lambda} I_{\mu,p} H + \sum_{r,t} c_{r,\mu,t} I_{r,\lambda} I_{\nu,t} B + (u_{1,x} + u_{3,z}; h_\lambda' h_\mu h_\nu) = 0 \tag{32$^{6)}$}$$

where the subscripts μ, λ and ν are $\mu = 1, 3, 5, \ldots$, $\lambda = 1, 2, 3, \ldots$ and $\nu = 1, 2, 3, \ldots$
With $h_\lambda(x) h_\mu(y) h_\nu'(z)$

$$\sum_{l,n} a_{l,\mu,n} I_{\lambda,l} I_{n,\nu} B + \sum_{p,q} b_{\lambda,p,q} I_{\mu,p} I_{q,\nu} L + c_{\lambda,\mu,\nu} \left(\nu \frac{\pi}{2H}\right)^2 L B H + (u_{1,x} + u_{3,z}; h_\lambda h_\mu h_\nu') = 0^{6)} \tag{33}$$

is obtained in an analogous manner, with $\mu = 1, 3, 5, \ldots$; $\lambda = 1, 2, 3, \ldots$ and $\nu = 1, 2, 3, \ldots$ The generalized product from equation (31) with $h_\lambda(x) h_\mu'(y) h_\nu(z)$ finally yields

$$\sum_{l,m} a_{l,m,\nu} I_{\lambda,l} I_{m\mu} H + b_{\lambda,\mu,\nu} \left(\mu \frac{\pi}{2B}\right)^2 L B H + \sum_{t,s} c_{\lambda,s,t} I_{s,\mu} I_{\nu,t} L + (u_{1,x} + u_{3z}; h_\lambda h_\mu' h_\nu) = 0 . \tag{34}$$

Shutting the throat means that $\bar{v}_{1\perp A3}$ and $\bar{v}_{3\perp A6}$ must vanish which causes $u_1(x) = 0$ and $u_3(z) = 0$. The consequences are $a_{\lambda,\mu,\nu} = 0$, $b_{\lambda,\mu,\nu} = 0$ and $c_{\lambda,\mu,\nu} = 0$. The vanishing of the coefficients $a_{\lambda,\mu,\nu}$, $b_{\lambda,\mu,\nu}$ and $c_{\lambda,\mu,\nu}$ has the consequence that $p_x' = 0$, $p_y' = 0$ and $p_z' + \varrho_B g = 0$. This is the hydrostatic limit case.

The coefficients $a_{l,m,n}$ can be interpreted as a three-dimensional matrix (figure 3). In this matrix each discrete point l, m, n is assigned to the coefficient $a_{l,m,n}$. The interpretation of $b_{o,p,q}$ and $c_{r,s,t}$ is analogous. Equation (32) can be thus interpreted that in a definite manner a well-defined coefficient in the a matrix is assigned to certain coefficients in the b and c matrices. An analogous interpretation is possible for the equations (33 and 34).

The equations (32 to 34) represent a system of $3 \cdot N^3$ equations for $3 \cdot N^3$ unknown quantities if the indices l, m, n, o, p, q, r, s and t vary between 1 and N.

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4) $(n_3; h_{\lambda,\mu,\nu}) = \left(\frac{1}{\eta} p_z'; h_{\lambda,\mu,\nu}\right) + \frac{1}{\eta} (\varrho_B g; h_{\lambda,\mu,\nu}) = \frac{1}{\eta} \int_0^{2H} \frac{d}{dz} K_{\lambda,\mu}(z) h_\nu(z) \, dz + \frac{1}{\eta} \varrho_B g (1; h_\lambda) (1; h_\mu) (1; h_\nu)$.

5) $h_l'(x) = \frac{d}{dx} h_l(x)$; $h_n'(z) = \frac{d}{dz} h_n(z)$; $f'_{m,n}(x) = \frac{d}{dx} f_{m,n}(x)$; $k'_{l,m}(z) = \frac{d}{dz} k_{l,m}(z)$.

6) $I_{j,\mu} = \frac{2 j \mu}{j^2 - \mu^2}$ for $j = 1, 3, 5, \ldots$; $\mu = 2, 4, 6, \ldots$; $I_{j,\mu} = 0$ for all other cases.

6. Mass current

The equations (32 to 34) in section 5. demonstrate that the velocity field $\underline{v}'$ is directly proportional to the mass current $\dot{M}$ because the coefficients in the series are determined by this quantity. For each $\dot{M}$ it is possible to calculate a velocity field from these equations. But only one $\dot{M}$ can exist in reality. Emanation equations are given in standard books of hydrodynamics only for ideal fluids. The main problem now is the determination of the mass current $\dot{M}$. One possibility to determine this quantity is by experiments. Such experiments, however, are not easy to carry out for melts.

In the sections 3. to 5. only the equation of motion and continuity was used. It is hoped to obtain further information from the energy-conservation law.

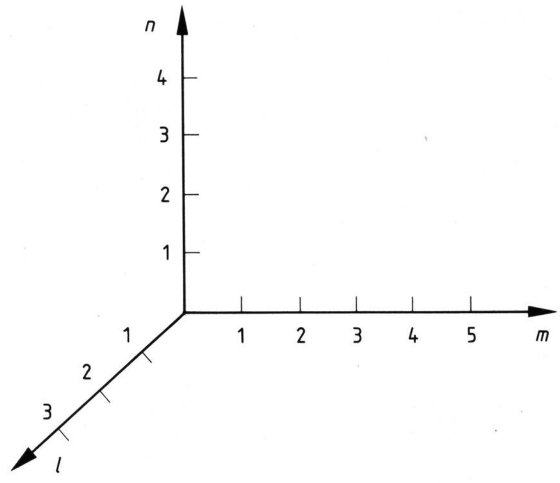
The fluid crossing the plane A_6 has a potential energy W_i and a mean kinetic energy W_{ki}. The analogous quantities in the plane A^* are W_a and W_{ka}.

The fluid flows from the top to the plane A^*. In this process the potential energy of the fluid decreases whereas the kinetic energy increases. The energy-conservation law demands that the decrease of potential energy must be equal to the increase of kinetic energy and friction energy W_f, thus equation (35) is obtained:

$$W_i - W_a = W_{ka} - W_{ki} + W_f \, . \tag{35}$$

With the equations (36 to 40)

$$dW_a = \dot{M}\, g\, H^*\, dt \, , \tag{36}$$



$$dW_i = \dot{M}\, g\, 2\, H\, dt \, , \tag{37}$$

$$dW_{ka} = \frac{1}{2}\, \dot{M}\, (\overline{v}_{1\perp A^*})^2\, dt = \frac{1}{2}\, \dot{M} \left(\frac{\dot{M}}{\varrho_B\, A^*}\right)^2 dt \, , \tag{38}$$

$$dW_{ki} = \frac{1}{2}\, \dot{M}\, (\overline{v}_{3\perp A6})^2\, dt = \frac{1}{2}\, \dot{M} \left(\frac{\dot{M}}{\varrho_B\, 2\, L\, 2\, B}\right)^2 dt \, , \tag{39}$$

$$dW_f = \dot{M}^2\, \dot{W}_f^*\, dt \, , \tag{40}$$

equation (41) is obtained

$$\frac{1}{2}\left[\frac{1}{\varrho_B^2\, A^{*2}} - \frac{1}{\varrho_B^2 16\, L^2\, B^2}\right] \dot{M}^2 + \dot{W}_f^*\, \dot{M} - g\,(2\,H - H^*) = 0 \, . \tag{41}$$

This is a quadratic equation determining $\dot{M}$ with the solution

$$\dot{M} = \delta^{-1}\left[-\dot{W}_f^* \pm \sqrt{(\dot{W}_f^*)^2 + 4\,\delta\, g\,(2\,H - H^*)}\right]^{7)} \tag{42}$$

where only the positive value of the root has a physical meaning. For ideal fluids ($\eta = 0$) $\dot{W}_f^*$ vanishes. The energy dissipated by friction is given by the equations (43 to 45)

$$W_f = \int \eta \triangle \underline{v}'\, d\underline{s}\, dV \, , \tag{43}$$

$$W_f = \int \eta\, (\triangle \underline{v}')\, \underline{v}'\, dt\, dV \, , \tag{44}$$

$$W_f = \eta\, \dot{M}^2 (\triangle \underline{v}^*)\, v^*\, dt\, dV = \dot{M}^2\, \dot{W}_f^*\, dt \tag{45}$$

where $\underline{v}' = \dot{M}\underline{v}^*$ and V = volume. As the velocity field $\underline{v}^*$ is known the quantity

$$\dot{W}_f^* = \eta \int_0^{2L} \int_0^{2B} \int_0^{2H} (\triangle \underline{v}^*)\, \underline{v}^*\, dx\, dy\, dz \tag{46}$$

can be evaluated.

7. Summary

Papers modelling flow fields in glass melting furnaces use numerical methods to solve the Navier-Stokes equations. It is the topic of this paper to present an analytical expression for the velocity field in the furnace. In a first step the fluid flow of an all-electric furnace at the mean temperature is discussed. The velocity field is obtained without information on the pressure field. The mass current determined by the straight-forward flow and caused by the gravity forces can be found out from the energy-conservation law. A following paper will deal with the convection flow in such furnaces.

[7]) $\delta = \frac{1}{2}\left[\frac{1}{\varrho_B^2\, A^{*2}} - \frac{1}{\varrho_B^2\, 16\, L^2\, B^2}\right]$.

8. Appendices

8.1. Pressure field

The pressure field is given by equation (47):

$$p'(x,y,z) = \sum_{\substack{m=1\\n=1}} F_{m,n}(x)\, h_m(y)\, h_n(z) + \sum_{\substack{l=1\\n=1}} G_{l,n}(y)\, h_l(x)\, h_n(z) + \sum_{\substack{l=1\\m=1}} K_{l,m}(z)\, h_l(x)\, h_m(y) \tag{47}$$

with the following equations for the different quantities:

$$h_l(x) = \sin\left(l \frac{\pi}{2\,L} x\right), \tag{48}$$

$$h_m(y) = \sin\left(m \frac{\pi}{2\,B} y\right), \tag{49}$$

$$h_n(z) = \sin\left(n \frac{\pi}{2\,H} z\right), \tag{50}$$

$$F_{m,n}(x) = A_{m,n} \sin h\,(c_{m,n}\, x) + A^*_{m,n} \cos h\,(c_{m,n}\, x), \tag{51}$$

$$G_{l,n}(y) = B_{l,n} \sin h\,(c_{l,n}\, y) + B^*_{l,n} \cos h\,(c_{l,n}\, x), \tag{52}$$

$$K_{l,m}(x) = D_{l,m} \sin h\,(c_{l,m}\, z) + D^*_{l,m} \cos h\,(c_{l,m}\, z), \tag{53}$$

$$c^2_{m,n} = \left(m \frac{\pi}{2\,B}\right)^2 + \left(n \frac{\pi}{2\,H}\right)^2, \tag{54}$$

$$c^2_{l,n} = \left(l \frac{\pi}{2\,L}\right)^2 + \left(n \frac{\pi}{2\,H}\right)^2, \tag{55}$$

$$c^2_{l,m} = \left(l \frac{\pi}{2\,L}\right)^2 + \left(m \frac{\pi}{2\,B}\right)^2. \tag{56}$$

In this solution unknowns are only the six groups of coefficients $A_{m,n}$, $A^*_{m,n}$, $B_{l,n}$, $B^*_{l,n}$, $D_{l,m}$ and $D^*_{l,m}$.

Because the plane $y = B$ is a symmetry plane the relation $p'(x, B + \bar{y}, z) = p'(x, B - \bar{y}, z)$ must be accomplished (figure 2). This means that the numbers m must be odd ($m = 1, 3, 5 \ldots$).

8.2. Velocity field

The field $u_3(x,y,z)$ is described by equation (57),

$$u_3(x,y,z) = \sum_{l,m} k_{l,m}(z)\, h_l(x)\, h_m(y) \tag{57}$$

with the following equations for the different quantities:

$$h_l(x) = \sin\left(l \frac{\pi}{2\,L} x\right), \tag{58}$$

$$h_m(y) = \sin\left(m \frac{\pi}{2\,B} y\right), \tag{59}$$

$$k_{l,m}(z) = e_{l,m} \sin h\,(c_{l,m}\, z) + e^*_{l,m} \cos h\,(c_{l,m}\, z), \tag{60}$$

$$c^2_{l,m} = \left(l \frac{\pi}{2\,L}\right)^2 + \left(m \frac{\pi}{2\,B}\right)^2, \tag{61}$$

$$\int_0^{2\,L} \int_0^{2\,B} u_3(x,y,0)\, h_l(x)\, h_m(y)\, \mathrm{d}x\, \mathrm{d}y = e^*_{l,m}\, L\, B, \tag{62}$$

$$\int_0^{2\,L} \int_0^{2\,B} u_3(x,y,2\,H)\, h_l(x)\, h_m(y)\, \mathrm{d}x\, \mathrm{d}y = \left[e_{l,m} \sin h\,(c_{l,m}\, 2\,H) + e^*_{l,m} \cos h\,(c_{l,m}\, 2\,H)\right] L\, B. \tag{63}$$

u_3 vanishes at the bottom thus $e^*_{l,m} = 0$ is yielded; u_3 at the top is determined by the mass current $\dot{M}$. With equation (64),

$$u_{3\perp A6} = -\frac{\dot{M}}{\varrho_B\, 4\, L\, B}, \tag{64}$$

equation (65) is obtained from equation (63)

$$-\frac{\dot{M}}{\varrho_B\, 4\, L\, B} (1; h_l)(1; h_m) = e_{l,m} \sin h(c_{l,m}\, 2\,H)\, L\, B \tag{65}$$

where $(1; h_l)$ and $(1; h_m)$ are given by

$$(1; h_l) = \int_0^{2\,L} h_l(x)\, \mathrm{d}x = -\frac{2\,L}{l\,\pi} [(-1)^l - 1], \tag{66}$$

$$(1; h_m) = \int_0^{2\,B} h_m(y)\, \mathrm{d}y = -\frac{2\,B}{m\,\pi} [(-1)^m - 1]. \tag{67}$$

Equation (65) determines the coefficients $e_{l,m}$. If $y = B$ is a symmetry plane, the number m must be odd ($m = 1, 3, 5 \ldots$).

The field $u_1(x,y,z)$ is described by equation (68)

$$u_1(x,y,z) = \sum_{\substack{m=1\\n=1}} f_{m,n}(x)\, h_m(y)\, h_n(z) \tag{68}$$

with the following equations for the different quantities:

$$h_m(y) = \sin\left(m \frac{\pi}{2B} y\right), \quad (69)$$

$$h_n(z) = \sin\left(n \frac{\pi}{2H} z\right), \quad (70)$$

$$f_{m,n}(x) = a_{m,n} \sin h\,(c_{m,n}\, x) + a^*_{m,n} \cos h\,(c_{m,n}\, x), \quad (71)$$

$$c^2_{m,n} = \left(m \frac{\pi}{2B}\right)^2 + \left(n \frac{\pi}{2H}\right)^2, \quad (72)$$

$$\int_0^{2B}\int_0^{2H} u_1(0,y,z)\, h_m(y)\, h_n(z)\, \mathrm{d}y\, \mathrm{d}z = a^*_{m,n}\, B\, H, \quad (73)$$

$$\int_0^{2B}\int_0^{2H} u_1(2L,y,z)\, h_m(y)\, h_n(z)\, \mathrm{d}y\, \mathrm{d}z = \left[a_{m,n} \sin h\,(c_{m,n}\, 2L) + a^*_{m,n} \cos h\,(c_{m,n}\, 2L)\right] B\, H. \quad (74)$$

u_1 vanishes for $x = 0$. This means $a^*_{m,n} = 0$. On A_3 u_1 vanishes without the throat. In the throat hold the following equations (75 to 77):

$$u_1 = \frac{\dot{M}}{\varrho_B A^*}, \quad (75)$$

with equation (75) thus equation (76) is obtained (figure 1),

$$\frac{\dot{M}}{\varrho_B A^*} = \int_{y_1}^{y_2}\int_0^{z_1} h_m(y)\, h_n(z)\, \mathrm{d}y\, \mathrm{d}z =$$

$$= a_{m,n} \sin h\,(c_{m,n}\, 2L)\, B\, H \quad (76)$$

or with $y_1 = B - B^*$, $y_2 = B + B^*$ and $z_1 = 2H^*$ equation (77) is obtained

$$\frac{\dot{M}}{\varrho_B A^*} \left[\frac{2B}{m\pi} \cos\left(m \frac{\pi}{2B} y\right)\right]_{B-B^*}^{B+B^*} \cdot \left[\frac{2H}{n\pi} \cos\left(n \frac{\pi}{2H} z\right)\right]_0^{2H^*} = a_{m,n} \sin h\,(c_{m,n}\, 2L)\, B\, H. \quad (77)$$

This equation determines the coefficients $a_{m,n}$ and $u_1(x,y,z)$ is known because the symmetry relative to the plane $y = B$, the number m must be odd ($m = 1, 3, 5, \ldots$).

9. Nomenclature

9.1. Symbols

a	matrix
A^*	area of throat
A_i	area of the furnace
b	matrix
$2B$	width of the furnace
c	matrix
$\underline{e}_i$	unit vector
$h_{l,m,n}$	Eigenfunktion of the Laplace operator
$2H$	height of the furnace
$I_{j,\mu}$	$I_{j,\mu} = 2j\mu/(j^2 - \mu^2)$
$2L$	length of the furnace
$\dot{M}$	mass current
$\underline{n}$	source term in equation (12) $\triangle \underline{v}' = -\underline{n}$
N	electric heating power
p	pressure
p'	pressure in the melt by the mean temperature
$p'_{\perp A_j}$	pressure on the area A_j
T	temperature
$\overline{T}$	mean value of the temperature
$\widetilde{T}$	temperature deviation ($\widetilde{T} = T - \overline{T}$)
T_B	reference temperature
$\underline{u}$	velocity field which satisfies the equation $\triangle \underline{u} = 0$ and the boundary conditions for the $\underline{v}'$ field ($\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$)
$\underline{v}$	velocity ($\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$)
$\overline{v}_i$	mean value of the velocity component v_i
$\overline{v}_{i\perp A_j}$	mean value of the velocity component v_i on the area A_j
$\underline{v}'$	velocity of the melt by the mean temperature ($\underline{v}' = \underline{u} + \underline{w}$)
$\underline{w}$	velocity field which satisfies the equation $\triangle \underline{w} = -\underline{n}$ and the boundary condition $w_{\perp A_i} = 0$ ($\underline{w} = w_1 \underline{e}_1 + w_2 \underline{e}_2 + w_3 \underline{e}_3$)
W_f	friction energy
$W_\mathrm{a}, W_\mathrm{i}$	potential energy
$W_\mathrm{ka}, W_\mathrm{ki}$	kinetic energy
x,y,z	coordinates
β	volume expansion coefficient of the melt
η	dynamic viscosity of the melt
ϱ_B	density of the melt by the reference temperature T_B

9.2. Indices

i,j, l,m,n, o,p,q, r,s,t, u,v, λ,μ,ν: integers if they are used as indices

10. References

[1] Ungan, A.: Three dimensional numerical modeling of glass melting process. Purdue University Lafayette, IN, Ph. D. thesis 1985.

[2] Harlow, F. H.; Welch, J. E.: Numerical calculation of time dependent viscous incompressible flow. Phys. Fluids **8** (1965) p. 2182–2189.

[3] Riederer, A.: Zur numerischen Berechnung und Verifikation dreidimensionaler Geschwindigkeits- und Temperaturfelder in elektrisch beheizten Glasschmelzwannen. Akad. Wiss., Zentralinstitut Mathematik und Mechanik (GDR), thesis 1978.

[4] Werner, U.-J.: Zwei Beiträge zur mathematischen Modellierung von Glasschmelzaggregaten mit vollelektrischer Beheizung. Hochschule Architektur und Bauwes., Fak. Naturwiss. Verfahrenstech., Weimar (GDR), thesis 1978.

[5] Grossmann, S.: Haben analytische Methoden in der Physik heute noch eine Chance? Phys. Bl. **45** (1989) no. 3, p. 76–81.

[6] Kneschke, A.: Differentialgleichungen und Randwertprobleme. Bd. 2. Partielle Differentialgleichungen. Leipzig: Teubner 1961.

[7] Oezisik, M. N.: Heat conduction. New York: Wiley 1980.

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