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Project title:Multidimensional moment problem and Schur algorithm

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1 General information

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2 Summary

The aim of the project was to study the multidimensional moment problem in terms of the multidimensional Stieltjes transform as the interpolation problem and construct the Schur algorithm for the truncated multidimensional moment problem.

The obtained results were related to classical and indefinite multidimensional moment problems and can also be applied to the multidimensional moment problem for atomic measure. Several step-by-step algorithms were obtained, which were based on different approaches. The first and basic was two-dimensional moment problem, one was studied separately from the multidimensional case. The truncated multidimensional moment problem was reformulated in terms of the multidimensional Stieltjes transform as the interpolation problem and also was studied.

The relation between one-dimensional and multidimensional moment problems was obtained. Hence, the obtained Schur algorithms for the multidimensional moment problem were base on the one-dimensional case. The description of all solution was found in terms P, J and S–fractions. The Padé approximants of the the truncated multidimensional moment problem were found. In additional, the description of all solutions of the full multidimensional moment problem and the indeterminate criterion for the full multidimensional moment problem were obtained.

Moreover, the Herglotz-Nevanlinna class functions in several variables was also studied. The asymptotic expansion of the Herglotz-Nevanlinna class functions in several variables was obtained. The corresponding interpolation problem in this class was formulated and solved.

All project goals were achieved.

3 Scientific progress report

The aim of the project was to study the multidimensional moment problem as the interpolation problem and construct (find) the Schur algorithm for the truncated multidimensional moment problem. It was possible because the multidimensional moment problem was reformulated in terms of the multidimensional Stieltjes transform as the interpolation problem.

Objectives: to study the two-dimensional moment problem as the interpolation moment problem, to construct the Schur algorithm for the two-dimensional moment and to describe all solution of the two-dimension moment problem in terms of continued fractions and to obtain criterions for these representation, to study the truncated multidimensional moment problem as the interpolation moment problem, to find and construct the step-by-step solution algorithm and to describe the all solution of the multidimensional moment problem in terms of continued fractions and to obtain criterions for these representation. To study the multidimensional moment problem with the atomic masses and to describe the set of the solutions using step-by-step algorithm, to solve the truncated multidimensional moment problem in the Herglotz-Nevanlinna classes functions in several variables as the the interpolation problem and to find the description of the all solution; to obtain Padé approximants of the the truncated multidimensional moment problem.

During the implementation of the project, all set goals were achieved. Briefly describe the content of the report. Subsection 3.1 contains the two-dimensional moment, the Schur algorithm for the two-dimensional moment and description of all solutions were obtained. The multidimensional moment problem was studied in Subsection 3.2, Subsection 3.3 and Subsection 3.4. Different research approaches were applied to these chapters. Hence, various step-by-step algorithms were obtained, the description of all solutions was found, Padé approximants were obtained. The Herglotz-Nevanlinna classes functions in several variables were studied in Subsection 3.5. The interpolation moment problem for the Herglotz-Nevanlinna function in several variables was formulated and solved.

3.1 Two-dimensional moment problem and Schur algorithm

In this subsection, we study a truncated two-dimensional moment problem in terms of the Stieltjes transform. The set of the solutions is described by the Schur step-by-step algorithm, which is based on the continued fraction expansion of the solutions. In particular, the obtained results are applicable to the two-dimensional moment problem for atomic measures.

The obtained results are published (see S. Kunis and I. Kovalyov, Two-dimensional moment problem and Schur algorithm, Integral Equations and Operator Theory, vol. 97, 5 (2025) doi:10.1007/s00020-024-02786-3).

3.2 Multidimensional moment problem and Schur algorithm

In the current subsection, the truncated multidimensional moment problem is studied in terms of the Stieltjes transform as the interpolation problem. A step-by-step algorithm is constructed for the multidimensional moment problem and the set of solutions is found in terms of continued fractions.

The obtained results are presented in the paper S. Kunis and I. Kovalyov, Multidimensional moment problem and Schur algorithm, (submitted).

Let us present the main results:

Preliminaries:

Let μ be a nonnegative Borel measure on $\mathbb{R}_+^n$, where $\sup(\mu) = A \subseteq \mathbb{R}_+^n$. The moment sequence $\mathbf{s} = \{s_{i_1,\dots,i_n}\}_{i_1,\dots,i_n=0}^{\ell}$ is defined by

$$s_{i_1,\dots,i_n} = \int_{\mathbb{R}_+^n} t_1^{i_1} \times \ldots \times t_n^{i_n} d\mu(t_1, \dots, t_n). \tag{3.1}$$

On the other hand, we can reformulated (3.1) in terms of the Stieltjes transform as an interpolation problem. Recall a $n-$variate Stieltjes transform (see [2])

$$\int_{\mathbb{R}_+^n} \frac{d\mu(t_1, t_2, ..., t_n)}{1 + \sum\limits_{i=1}^{n} t_i \tilde{z}_i}. \tag{3.2}$$

Setting $\tilde{z}_i = -\dfrac{1}{z_i}$, we define an associated function F by

$$F(z_1, z_2, ..., z_n) = -\frac{1}{\prod\limits_{i=1}^{n} z_i} \int_{\mathbb{R}_+^n} \frac{d\mu(t_1, t_2, ..., t_n)}{1 - \sum\limits_{i=1}^{n} \frac{t_i}{z_i}}, \quad \text{where } z_i \widehat{\to} \infty. \tag{3.3}$$

Moreover, (3.3) can be rewritten in terms of the Stieltjes transform as

$$\begin{aligned} F(z_1, z_2, ..., z_n) &= -\frac{1}{\prod\limits_{i=1}^{n} z_i} \int_{\mathbb{R}_+^n} \frac{d\mu(t_1, t_2, ..., t_n)}{1 - \sum\limits_{i=1}^{n} \frac{t_i}{z_i}} = \\ &- \sum_{k=0}^{\infty} \sum_{\substack{\alpha_i^k \geq 0 \\ \alpha_1^k + \ldots \alpha_n^k = k}} \binom{k}{\alpha_1^k, \ldots, \alpha_n^k} \frac{s_{\alpha_1^k, \alpha_2^k, \ldots, \alpha_n^k}}{z_1^{\alpha_1^k+1} z_2^{\alpha_2^k+1} \ldots z_n^{\alpha_n^k+1}}. \end{aligned} \tag{3.4}$$

Furthermore, we can study the truncated moment problem for atomic measure. Let μ be an atomic measure on $\mathbb{R}^n$, where $\sup(\mu) = A \subseteq \mathbb{R}^n$ and

$$\mu(x) = \sum_{k=0}^{M} m_k \delta_{x_k}(x), \quad m_k > 0. \tag{3.5}$$

In this case, the moment sequence $\mathbf{s} = \{s_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n=0}^{\ell}$ is defined by

$$s_{i_1,\ldots,i_n} = \int_{\mathbb{R}^n} x_1^{i_1} \times \ldots \times x_n^{i_n} d\mu(x_1, ..., x_n) = \sum_{k=0}^{M} m_k t_1^{i_1} \times \ldots \times t_n^{i_n} \tag{3.6}$$

and (3.2)–(3.4) hold.

However, we can reformulate the moment problem (3.1) in terms of the generalized moments. Define a linear functional $\mathfrak{S}$ on the monomials $t_1^{\alpha_1} \times \ldots \times t_n^{\alpha_n}$ by

$$\mathfrak{S}(t_1^{\alpha_1} \times \ldots \times t_n^{\alpha_n}) = s_{\alpha_1,\ldots,\alpha_n}. \tag{3.7}$$

In the general case, the associated function can be defined by

$$\begin{aligned} F(z_1, ..., z_n) &= -\frac{1}{\prod\limits_{i=1}^{n} z_i} \mathfrak{S}\left(\frac{1}{1 - \sum\limits_{i=1}^{n} \frac{t_i}{z_i}} \right) = -\sum_{k=0}^{n\times\ell} \sum_{\substack{\alpha_i^k \geq 0 \\ \alpha_1^k + \ldots + \alpha_n^k = k}} \binom{k}{\alpha_1^k, \ldots, \alpha_n^k} \frac{s_{\alpha_1^k, \alpha_2^k, \ldots, \alpha_n^k}}{z_1^{\alpha_1^k+1} \ldots z_n^{\alpha_n^k+1}} + \\ &+ o\left(\sum_{\substack{\alpha_i^{n\times\ell} \geq 0 \\ \alpha_1^{n\times\ell} + \ldots + \alpha_n^{n\times\ell} = n\times\ell}} \frac{1}{z_1^{\alpha_1^{n\times\ell}+1} \ldots z_n^{\alpha_n^{n\times\ell}+1}} \right). \end{aligned} \tag{3.8}$$

Moreover, (3.8) can be rewritten in the symmetric form as follows:

$$F(z_1,...,z_n) = -\frac{\mathfrak{s}_0}{\prod\limits_{j=1}^{n} z_j} - \frac{\mathfrak{s}_1(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^2} - \ldots - -\frac{\mathfrak{s}_\ell(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^{\ell+1}} + o\left(\frac{1}{\prod\limits_{j=1}^{n} z_j^{\ell+1}}\right), \tag{3.9}$$

where

$$\mathfrak{s}_0 = s_{0,\ldots 0}, \quad \mathfrak{s}_j(z_1,...,z_n) = \sum_{\substack{\alpha_i \geq 0,\\ \text{at least one}\\ \alpha_i = j}}^{j} \binom{\sum_{i=1}^{n} \alpha_i}{\alpha_1,\ldots,\alpha_n} s_{\alpha_1,\ldots,\alpha_n} \prod_{i=1}^{n} z_i^{j-\alpha_i}. \tag{3.10}$$

The sequence $\mathfrak{s} = \{\mathfrak{s}_j\}_{j=0}^{\ell}$ is called an associated sequence of the sequence $\mathbf{s} = \{s_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n=0}^{\ell}$ and F is called an associated function.

Definition3.1. Let $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\ldots,z_n)\}_{j=0}^{\ell}$ be an associated sequence of the sequence of real numbers $\mathbf{s} = \{s_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n=0}^{\ell}$ such that (3.10) holds and let $\mathfrak{n} = \{\mathfrak{n}_i | \mathfrak{n}_i \in \mathbb{N}$ and $1 \leq \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$. $\mathfrak{N}_\mathfrak{n}(s)$ is called a regular set of the sequence $\mathfrak{s}$ defined by

$$\mathfrak{N}_\mathfrak{n}(s) = \{(z_1,\ldots,z_n)| \quad D_{\mathfrak{n}_i}(z_1,\ldots,z_n) \neq 0 \quad \text{for all} \quad i = \overline{1,N}\}, \tag{3.11}$$

where

$$D_i(z_1,\ldots,z_n) = \begin{vmatrix} \mathfrak{s}_0(z_1,\ldots,z_n) & \ldots & \mathfrak{s}_{i-1}(z_1,\ldots,z_n) \\ \vdots & \ddots & \vdots \\ \mathfrak{s}_{i-1}(z_1,\ldots,z_n) & \ldots & \mathfrak{s}_{2i-2}(z_1,\ldots,z_n) \end{vmatrix}. \tag{3.12}$$

Moreover, $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\ldots,z_n)\}_{j=0}^{\ell}$ is called a regular sequence on the set $\mathfrak{N}_\mathfrak{n}(s)$.

Hence, we can formulate the truncated multidimensional moment problem $\mathbf{MP}(\mathbf{s}, \mathfrak{N}_\mathfrak{n}(s))$:

Given a sequence of real numbers $\mathbf{s} = \{s_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n=0}^{\ell}$ and a set $\mathfrak{n} = \{\mathfrak{n}_i|\mathfrak{n}_i \in \mathbb{N}$ and $1 \leq \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$ such that $\ell \geq 2n_N - 1$ and the associated sequence $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\ldots,z_n)\}_{j=0}^{2\mathfrak{n}_N-1}$ defined by (3.10) is the regular on the set $\mathfrak{N}_\mathfrak{n}(s)$. Describe the set of functions F, which admit the asymptotic expansion

$$F(z_1,...,z_n) = -\frac{\mathfrak{s}_0}{\prod\limits_{j=1}^{n} z_j} - \frac{\mathfrak{s}_1(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^2} - \ldots - -\frac{\mathfrak{s}_\ell(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^{\ell+1}} + o\left(\frac{1}{\prod\limits_{j=1}^{n} z_j^{\ell+1}}\right). \tag{3.13}$$

Theorem 3.1. Let $\mathbf{s} = \{s_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n=0}^{\ell}$ be a sequence of real numbers and let $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\ldots,z_n)\}_{j=0}^{\ell}$ be its associated sequence defined by (3.10) such that $\mathfrak{s}$ is the regular sequence on the set $\mathfrak{N}_\mathfrak{n}(s)$, where $\mathfrak{n} = \{\mathfrak{n}_i|\mathfrak{n}_i \in \mathbb{N}$ and $1 \leq \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$ and $\ell \geq 2\mathfrak{n}_N - 1$. Then any solution of the truncated problem $\mathbf{MP}(\mathbf{s}, \mathfrak{N}_\mathfrak{n}(s))$ admits the representation

$$F(z_1,...,z_n) = -\cfrac{b_0(z_1,...,z_n)}{a_0(z_1,...,z_n) - \cfrac{b_1(z_1,...,z_n)}{a_1(z_1,...,z_n) - \cdots - \cfrac{b_{N-1}(z_1,...,z_n)}{a_{N-1}(z_1,...,z_n) + \tau(z_1,...,z_n)}}}, \tag{3.14}$$

where the parameter τ adits the following asymptotic expansion

$$\tau(z_1,...,z_n) = -\frac{\mathfrak{s}_0^{(N)}}{\prod\limits_{j=1}^{n} z_j} - \frac{\mathfrak{s}_1^{(N)}(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^2} - \ldots - -\frac{\mathfrak{s}_{\ell-2n_N}^{(N)}(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^{\ell-2n_N+1}} + o\left(\frac{s_{\ell-2n_N}^{(N)}(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^{\ell-2n_N+1}}\right), \tag{3.15}$$

the atoms (a_j, b_j) can be found by

$$\begin{aligned}
&b_0(z_1,...,z_n) = \mathfrak{s}_{n_1-1}(z_1,...,z_n) \quad \text{and} \quad b_j = \mathfrak{s}^{(j)}_{\nu-1}(z_1,...,z_n),\\
&a_j(z_1,...,z_n) = \frac{1}{D^{(j-1)}_{\nu}(z_1,...,z_n)}\begin{vmatrix} \mathfrak{s}^{(j-1)}_0(z_1,...,z_n) & \mathfrak{s}^{(j-1)}_1(z_1,...,z_n) & \dots & \mathfrak{s}^{(j-1)}_{\nu}(z_1,...,z_n)\\ \dots & \dots & \dots & \dots \\ \mathfrak{s}^{(j-1)}_{\nu-1}(z_1,...,z_n) & \mathfrak{s}^{(j-1)}_{\nu}(z_1,...,z_n) & \dots & \mathfrak{s}^{(j-1)}_{2\nu-1}(z_1,...,z_n)\\ 1 & \prod\limits_{j=1}^{n} z_j & \dots & \prod\limits_{j=1}^{n} z_j^{\nu} \end{vmatrix},\\
&\mathfrak{s}^{(j)}_i(z_1,...,z_n) = \frac{(-1)^{i+\nu}}{\left(\mathfrak{s}^{(j-1)}_{\nu-1}(z_1,...,z_n)\right)^{i+\nu+2}}\begin{vmatrix} \mathfrak{s}^{(j-1)}_{\nu}(z_1,...,z_n) & \mathfrak{s}^{(j-1)}_{\nu-1}(z_1,...,z_n) & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ \vdots & & & \ddots & \mathfrak{s}^{(j-1)}_{\nu-1}(z_1,...,z_n)\\ \mathfrak{s}^{(j-1)}_{2\nu+i}(z_1,...,z_n) & \dots & \dots & \dots & \mathfrak{s}^{(j-1)}_{\nu}(z_1,...,z_n) \end{vmatrix},
\end{aligned} \tag{3.16}$$

where $i = \overline{0, \ell - 2n_j}$, $j = \overline{1, N}$, $\mathfrak{s}^{(0)}_i(z_1,...,z_n) = \mathfrak{s}_i(z_1,...,z_n)$, $\nu = n_j - n_{j-1}$ and $n_0 = 0$.

Proposition3.1. Let $\mathbf{s} = \{s_{i_1,...,i_n}\}_{i_1,...,i_n=0}^{\ell}$ be a sequence of real numbers and let $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\dots,z_n)\}_{j=0}^{\ell}$ be its associated sequence defined by (3.10) such that $\mathfrak{s}$ is the regular sequence on the set $\mathfrak{N}_{\mathfrak{n}}(s)$, where $\mathfrak{n} = \{\mathfrak{n}_i | \mathfrak{n}_i \in \mathbb{N} \text{ and } 1 \le \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$ and $\ell \ge 2\mathfrak{n}_N - 1$. Then any solution of the truncated problem $\mathbf{MP}(\mathbf{s}, \mathfrak{N}_{\mathfrak{n}}(s))$ can be represented as

$$F(z_1,...,z_n) = \frac{Q_{\mathfrak{n}_{N-1}}(z_1,...,z_n)\tau(z_1,...,z_n) + Q_{\mathfrak{n}_N}(z_1,...,z_n)}{P_{\mathfrak{n}_{N-1}}(z_1,...,z_n)\tau(z_1,...,z_n) + P_{\mathfrak{n}_N}(z_1,...,z_n)}, \tag{3.17}$$

where the parameter τ satisfies (3.15), $P_{\mathfrak{n}_j}$ and $Q_{\mathfrak{n}_j}$ are the solutions of the following system

$$\begin{aligned}
&b_j(z_1,...,z_n)y_{\mathfrak{n}_{j-1}} - a_j(z_1,...,z_n)y_{\mathfrak{n}_j} + y_{\mathfrak{n}_{j+1}} = 0, \quad j = \overline{0, N-1},\\
&P_{\mathfrak{n}_{-1}}(z_1,...,z_n) \equiv 0,\ P_{\mathfrak{n}_0}(z_1,...,z_n) \equiv 1 \text{ and } Q_{\mathfrak{n}_{-1}}(z_1,...,z_n) \equiv 1,\ Q_{\mathfrak{n}_0}(z_1,...,z_n) \equiv 0,
\end{aligned} \tag{3.18}$$

the atoms (a_j, b_j) can be found by (3.16).

Next, we study the truncated multidimensional moment problem $\mathbf{MP}(\mathbf{s})$ and we find a description of the solutions in terms of $\mathbf{S}-$ fraction. First of all, we introduce a plus regular set. It will be an important object of our investigation. Since $\mathbf{MP}(\mathbf{s})$ will be studied on this set.

Definition3.2. Let $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\dots,z_n)\}_{j=0}^{\ell}$ be an associated sequence of the sequence of real numbers $\mathbf{s} = \{s_{i_1,...,i_n}\}_{i_1,...,i_n=0}^{\ell}$ such that (3.10) holds and let $\mathfrak{n} = \{\mathfrak{n}_i | \mathfrak{n}_i \in \mathbb{N} \text{ and } 1 \le \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$. $\mathfrak{N}^+_{\mathfrak{n}}(s)$ is called a plus regular set of the sequence $\mathfrak{s}$, which is defined by

$$\mathfrak{N}^+_{\mathfrak{n}}(s) = \{(z_1,\dots,z_n) |\, D_{\mathfrak{n}_i}(z_1,\dots,z_n) \ne 0 \text{ and } D^+_{\mathfrak{n}_i}(z_1,\dots,z_n) \ne 0 \text{ for all } \quad i = \overline{1,N}\}, \tag{3.19}$$

where D_i is defined by (3.12) and

$$D^+_i(z_1,\dots,z_n) = \begin{vmatrix} \mathfrak{s}_1(z_1,\dots,z_n) & \dots & \mathfrak{s}_i(z_1,\dots,z_n)\\ \vdots & \ddots & \vdots\\ \mathfrak{s}_i(z_1,\dots,z_n) & \dots & \mathfrak{s}_{2i-1}(z_1,\dots,z_n) \end{vmatrix}. \tag{3.20}$$

Moreover, $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\dots,z_n)\}_{j=0}^{\ell}$ is called a plus regular sequence on the set $\mathfrak{N}^+_{\mathfrak{n}}(s)$.

Hence, we can formulate the truncated multidimensional moment problem $\mathbf{MP}(\mathbf{s}, \mathfrak{N}^+_{\mathfrak{n}}(s))$:

Given a sequence of real numbers $\mathbf{s} = \{s_{i_1,...,i_n}\}_{i_1,...,i_n=0}^{\ell}$ and a set $\mathfrak{n} = \{\mathfrak{n}_i | \mathfrak{n}_i \in \mathbb{N} \text{ and } 1 \le \mathfrak{n}_1 < \mathfrak{n}_2 < \ldots < \mathfrak{n}_N,\ i = \overline{1,N}\}$ such that $\ell \ge 2\mathfrak{n}_N - 1$ and the associated sequence $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\dots,z_n)\}_{j=0}^{2\mathfrak{n}_N-1}$ defined by (3.10) is the plus regular on the set $\mathfrak{N}^+_{\mathfrak{n}}(s)$. Describe the set of functions F, which admit the asymptotic expansion

$$F(z_1,...,z_n) = -\frac{\mathfrak{s}_0}{\prod\limits_{j=1}^{n} z_j} - \frac{\mathfrak{s}_1(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^2} - \ldots - -\frac{\mathfrak{s}_\ell(z_1,...,z_n)}{\prod\limits_{j=1}^{n} z_j^{\ell+1}} + o\left(\frac{1}{\prod\limits_{j=1}^{n} z_j^{\ell+1}}\right). \tag{3.21}$$

Theorem 3.2. Let $\mathbf{s} = \{s_{i_1,\dots,i_n}\}_{i_1,\dots,i_n=0}^{2\mathfrak{n}_N-1}$ be a sequence of real numbers and let $\mathfrak{s} = \{\mathfrak{s}_j(z_1,\dots,z_n)\}_{j=0}^{2\mathfrak{n}_N-1}$ be its associated sequence defined by (3.10) such that $\mathfrak{s}$ is the plus regular sequence on the set $\mathfrak{N}_{\mathfrak{n}}^{+}(s)$, where $\mathfrak{n} = \{\mathfrak{n}_i | \mathfrak{n}_i \in \mathbb{N}$ and $1 \le \mathfrak{n}_1 < \mathfrak{n}_2 < \dots < \mathfrak{n}_N,\ i = \overline{1,N}\}$. Then any solution of the truncated problem $\mathbf{MP}(\mathbf{s}, \mathfrak{N}_{\mathfrak{n}}^{+}(s))$ admits the representation

$$F(z_1,...,z_n) = \cfrac{1}{-\prod\limits_{j=1}^{n} z_j m_1(z_1,...,z_n) + \cfrac{1}{l_1(z_1,...,z_n) + \cdots + \cfrac{1}{-\prod\limits_{j=1}^{n} z_j m_N(z_1,...,z_n) + \cfrac{1}{l_N(z_1,...,z_n) + \tau(z_1,...,z_n)}}}}, \tag{3.22}$$

where the parameter τ adits the following asymptotic expansion

$$\tau(z_1,...,z_n) = o\left(l_N(z_1,...,z_n)\right) \tag{3.23}$$

the atoms (m_j, l_j) can be calculated in terms of the atoms (a_j, b_j) as follows

$$\begin{aligned} &b_0(z_1,...,z_n) = \frac{1}{d_0(z_1,...,z_n)} \quad \text{and} \quad a_0(z_1,...,z_n) = \frac{1}{d_0(z_1,...,z_n)}\left(\prod_{i=1}^{n} z_i m_1(z_1,...,z_n) - \frac{1}{l_1(z_1,...,z_n)}\right),\\ &b_{j-1}(z_1,...,z_n) = \frac{1}{l_{j-1}^2(z_1,...,z_n) d_{j-1}(z_1,...,z_n) d_j(z_1,...,z_n)},\\ &a_{j-1}(z_1,...,z_n) = \frac{1}{d_j(z_1,...,z_n)}\left(\prod_{i=1}^{n} z_i m_j(z_1,...,z_n) - \frac{1}{l_{j-1}(z_1,...,z_n)} - \frac{1}{l_j(z_1,...,z_n)}\right). \end{aligned} \tag{3.24}$$

Furthermore, (3.22) can be rewritten as

$$F(z_1,...,z_n) = \frac{Q_{2N}^{+}(z_1,...,z_n) + Q_{2N-1}^{+}(z_1,...,z_n)\tau(z_1,...,z_n)}{P_{2N}^{+}(z_1,...,z_n) + P_{2N-1}^{+}(z_1,...,z_n)\tau(z_1,...,z_n)}, \tag{3.25}$$

where P_j^{+} and Q_j^{+} can be found as solutions of the following system

$$\begin{cases} y_{2i-1} - y_{2i-3} = -\prod\limits_{i=1}^{n} z_i m_i(z_1,...,z_n) y_{2i-2}, \\ y_{2i} - y_{2i-2} = l_i(z_1,...,z_n) y_{2i-1}, \end{cases} \qquad i = \overline{1,N}, \tag{3.26}$$

subject to the initial conditions

$$P_{-1}^{+}(z_1,...,z_n) \equiv 0,\ P_0^{+}(z_1,...,z_n) \equiv 1,\ Q_{-1}^{+}(z_1,...,z_n) \equiv 1 \text{ and } Q_0^{+}(z_1,...,z_n) \equiv 0. \tag{3.27}$$

3.3 Multidimensional moment problem and diagonal Schur algorithm

The multidimensional moment problem is studied in terms of the Steiltjes transform. The diagonal step-by-step algorithm is constructed for the multidimensional moment problem. The set of solutions of the full multidimensional moment problem is found in terms of the continued fractions. Moreover, the diagonal step-by-step algorithm can be applied to the special truncated multidimensional moment problem. There are additional results.

The obtained results are presented in the paper: I. Kovalyov, Multidimensional moment problem and diagonal step-by-step algorithm, https://arxiv.org/abs/2501.05853.

3.4 Multidimensional moment problem and Stieltjes transform

The truncated multidimensional moment problem is studied in terms of the Stieltjes transform as the interpolation problem. A step-by-step algorithm is constructed for the truncated and full multidimensional moment problems and the set of solutions is found in terms of continued fractions. Padé approximants are found. There are additional results.

The obtained results are presented in the paper: I. Kovalyov, Multidimensional moment problem and Stieltjes transform, https://arxiv.org/abs/2501.05860.

3.5 Herglotz-Nevanlinna functions in several variables, asymptotics and Stieltjes transform

In this subsection, we study the indefinite moment problem associated with the Herglotz-Nevanlinna functions in several variables.

The obtained results are presented in the paper: I. Kovalyov, Herglotz-nevanlinna functions in several variables, asymptotics and Stieltjes transform (submitted).

Let us present the main results:

Theorem 3.3.([13]) A function $f : \mathbb{C}^{+n} \to \mathbb{C}$ is a Herglotz-Nevanlinna function in several variables if and only if f can be rewritten as

$$f(z) = \mathfrak{a} + \sum_{k=1}^{n} \mathfrak{b}_k z_k + \frac{1}{\pi^n}\int_{\mathbb{R}^n} K_n(z,t)d\mu(t), \tag{3.28}$$

where $\mathfrak{a} \in \mathbb{R}$, $\mathfrak{b}_k \geq 0$ and μ is a positive Borel measure on $\mathbb{R}^n$ satisfying the growth condition

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} \frac{1}{(t_k^2+1)} d\mu(t) < \infty, \tag{3.29}$$

the kernel function K_n is defined by

$$K_n(z,t) = i\left(\frac{2}{(2i)^n}\prod_{k=1}^{n}\left(\frac{1}{t_k - z_k} - \frac{1}{t_k+i}\right) + \frac{1}{(2i)^n}\prod_{k=1}^{n}\left(\frac{1}{t_k - i} - \frac{1}{t_k+i}\right)\right), \tag{3.30}$$

where $z \in \mathbb{C}^{+n}$ and $t \in \mathbb{R}^n$. Moreover, the following Nevanlinna condition holds

$$\sum_{\substack{\rho\in\{-1,0,1\}^n \\ -1\in\rho \wedge 1\in\rho}} \int_{\mathbb{R}^n} N_{\rho_1,1}\dots N_{\rho_n,n} d\mu(t) = 0 \tag{3.31}$$

for all $z \in \mathbb{C}^{+n}$ and

$$N_{-1,j} = \frac{1}{t_j - z_j} - \frac{1}{t_j - i}, \quad N_{0,j} = \frac{1}{t_j - z_j} - \frac{1}{t_j + i} \text{ and } N_{1,j} = \frac{1}{t_j + i} - \frac{1}{t_j - \overline{z}_j}. \tag{3.32}$$

Lemma 3.1. Let f be a Herglotz-Nevanlinna function in n variables, and let (3.28) and $\mu(t) = \prod_{k=1}^{n} \mu_k(t_k)$ hold. Then f admits the following asymptotic expansion

$$\begin{aligned} f(z) = \mathfrak{a} + \sum_{k=1}^{n} \mathfrak{b}_k z_k + \frac{i}{\pi^n}\int_{\mathbb{R}^n}\prod_{k=1}^{n}\frac{1}{t_k^2+1}d\mu(t) + \\ + \frac{1}{\pi^n(2i)^{n-1}}\prod_{k=1}^{n}\left(-\frac{s_0^{(k)}}{z_k} - \frac{s_1^{(k)}}{z_k^2} - \dots - \frac{s_{\ell_k}^{(k)}}{z_k^{\ell_k+1}} + o\left(\frac{1}{z_k^{\ell_k+1}}\right) - \int_{\mathbb{R}}\frac{d\mu_k(t_k)}{t_k+i}\right), \quad z\widehat{\to}\infty, \end{aligned} \tag{3.33}$$

where the sequence $\mathbf{s}^{(k)} = \{s_j^k\}_{j=0}^{\ell_k}$ is associated with the measure μ_k and one can be found by

$$s_j^{(k)} = \int_{\mathbb{R}} t_k^j d\mu_k(t_k), \qquad j = \overline{0, \ell_k}. \tag{3.34}$$

Theorem 3.4. Let f be a Herglotz-Nevanlinna function in n variables and let $\mu(t) = \prod_{k=1}^{n} \mu_k(t_k)$ in (3.28) with (3.29). Let $\mathbf{s}^{(k)} = \{s_j^k\}_{j=0}^{2N_k-1}$ be associated with the measure μ_k for all $k = \overline{1,n}$, and let $\mathcal{N}(\mathbf{s}^{(k)}) = \left\{n_j^{(k)}\right\}_{j=1}^{N_k}$ be the set of normal indices of $\mathbf{s}^{(k)}$. Then f admits the following asymptotic expansion

$$f(z) = \mathfrak{a} + \sum_{k=1}^{n} \mathfrak{b}_k z_k + A + \frac{1}{\pi^n (2i)^{n-1}} \prod_{k=1}^{n} \left(\mathop{\mathbf{K}}_{j=0}^{N_k-1} \left(-\frac{b_j^{(k)}}{a_j^{(k)}(z_k)} \right) - B_k \right), \; z \widehat{\to} \infty, \tag{3.35}$$

where the atoms $(a_j^{(k)}, b_j^{(k)})$ are associated with the sequence $\mathbf{s}^{(k)} = \{s_j^k\}_{j=0}^{2N_k-1}$ for all $k = \overline{1,n}$ and τ_k satisfy the following

$$\begin{cases} \tau_k \dfrac{\prod_{j=0}^{m} b_{2j}^{(k)}}{\prod_{j=0}^{m} b_{2j+1}^{(l)}} \in \mathbf{N}_{\kappa^{(k)} - \kappa^{(k)}_{N_k-1}}, \text{ if } N_k - 1 = 2m+1; \\ \tau_k \dfrac{\prod_{j=0}^{m} b_{2j-1}^{(k)}}{\prod_{j=0}^{m} b_{2j}^{(k)}} \in \mathbf{N}_{\kappa^{(k)} - \kappa^{(k)}_{N_k-1}}, \text{ if } N_k - 1 = 2m, \end{cases}$$

$$\kappa^{(k)}_{N_k-1} = \sum_{j=0}^{N_k-1} \tilde{\kappa}_j^{(k)}, \; \tilde{\kappa}_{2m}^{(k)} = \kappa_-\left(a_{2m}^{(k)} \frac{\prod_{j=0}^{m} b_{2j-1}^{(k)}}{\prod_{j=0}^{m} b_{2j}^{(k)}} \right), \; \tilde{\kappa}_{2m+1}^{(k)} = \kappa_-\left(a_{2m+1}^{(k)} \frac{\prod_{j=0}^{m} b_{2j}^{(k)}}{\prod_{j=0}^{m} b_{2j+1}^{(k)}} \right), \tag{3.36}$$

$$\tau_k(z_k) = o(1), \quad z_k \widehat{\to} \infty \quad \text{for } k = \overline{1,n}.$$

Furthermore, (3.35) can be rewritten in terms of the polynomials of the first and second kind as

$$f(z) = \mathfrak{a} + \sum_{k=1}^{n} \mathfrak{b}_k z_k + A + \frac{1}{\pi^n (2i)^{n-1}} \prod_{k=1}^{n} \left(-\frac{Q^{(k)}_{n_{N_k-1}}(z_k)\tau_k(z_k) + Q^{(k)}_{n_{N_k}}(z_k)}{P^{(k)}_{n_{N_k-1}}(z_k)\tau_k(z_k) + P^{(k)}_{n_{N_k}}(z_k)} - B_k \right), \; z \widehat{\to} \infty, \tag{3.37}$$

where $P_j^{(k)}$ and $Q_j^{(k)}$ are the polynomials of the first and second kind associated with the sequence $\mathbf{s}^{(k)} = \{s_j^k\}_{j=0}^{2N_k-1}$, respectively.

Remark3.1. If μ_k is positive, then

$$\int_{\mathbb{R}} \frac{d\mu_k(t_k)}{t_k - z_k} \quad \text{and} \quad \tau_k$$

are Nevanlinna functions.

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4 Published Project Results

4.1 Publications with scientific quality assurance

1. S. Kunis and I. Kovalyov, Two-dimensional moment problem and Schur algorithm, Integral Equations and Operator Theory, vol. 97, 5 (2025) doi:10.1007/s00020-024-02786-3, (open access is available on the preprint server https://arxiv.org/abs/2404.03342).

4.2 Other publications and published result

Another papers are submitted to peer-reviewed journals, but these papers have not yet been published. However, some papers are published on preprint servers (it is indicated):
1. I. Kovalyov, Multidimensional moment problem and Stieltjes transform (under review), https://arxiv.org/abs/2501.05860, (open access).
2. I. Kovalyov, Multidimensional moment problem and diagonal step-by-step algorithm (under review), https://arxiv.org/abs/2501.05853, (open access).
3. S. Kunis and I. Kovalyov, Multidimensional moment problem and Schur algorithm, (submitted).
4. I. Kovalyov, Herglotz-nevanlinna functions in several variables, asymptotics and Stieltjes transform (submitted).