COmPwise: Computing Optimal Piecewise Linear Functions and Their Applications
Final Report

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Public Part

1 General Information

- DFG Reference Number: RE 4249/2-1
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2 Summary

English (2.998 Characters) Piecewise linear (PWL) functions consist out of affine segments which intersect at breakpoints. They are used to fit a set of discrete data points or to approximate non-linear functions. PWL functions can be modeled by mixed-integer linear programming (MILP) techniques, where binary variables are used to assign segments. When solving large scale mixed-integer non-linear programming (MINLP) problems, PWL functions can be used to approximate the non-linearities, leading to a MILP model; this can significantly enhance computational tractability.

The goals of this project are to (1) advance the state-of-the-art in models for PWL function fitting and applications thereof, (2) develop tailored solution algorithms and (3) present a non-convex nested Benders decomposition algorithm for MINLP problems based on PWL approximations.

Regarding the first goal, a comparison of two existing MILP model is performed, showcasing the superiority of one model from a theoretical and experimental perspective across a number of different benchmarks. A framework for a class of problems using PWL functions,

including regression and clustering as well as feature selection is proposed. This framework is extended to support vector machines. Finally, a model from the literature on PWL computations is investigated and three enhancements are proposed.

Regarding the second goal, a tailored combinatorial Benders decomposition for PWL function computation with inbuilt outlier detection is presented which showcases speedups of significant magnitude. Further, an efficient method for approximating non-linear continuous functions using PWL functions, showcasing speedups of up to 100,000 times compared to the state-of-the-art, is developed. To our knowledge, this is currently the fastest approach for univariate PWL function fitting; a number of benchmark problems have been solved for the first time. An R-package has been released. Finally, a spatial branch-and-bound algorithm is developed to efficiently solve optimization problems containing univariate PWL functions.

Regarding the third goal, a multi-layered decomposition framework, the so-called non-convex nested Benders decomposition, allowing to solve multi-stage (stochastic) MINLP problems with proven exactness is proposed. It uses PWL functions to approximate the MINLP by an MILP, and this approximation is dynamically refined if required. To solve the MILP, it is decomposed and certain Lagrangian dual problems are solved to iteratively generate non-convex approximations of the arising value functions. As this is computationally challenging, some alternative strategies for this step, which are either more efficient or yield cuts with favorable properties (e.g. Pareto-optimality) are explored. To broaden the theoretical understanding of the concepts used in this new framework, among them Lipschitz regularization and cut generation, their mathematical properties and interdependencies are analyzed and deep connections are revealed.

German (2.995 Characters) Stückweise lineare (PWL) Funktionen bestehen aus affinen Segmenten, die sich an Knickpunkten schneiden. Sie dienen dazu, diskrete Datenpunkte zu approximieren oder nichtlineare Funktionen näherungsweise darzustellen. PWL-Funktionen lassen sich mit gemischt-ganzzahliger linearer Programmierung (MILP) modellieren, wobei binäre Variablen zur Segmentzuordnung verwendet werden. Bei großskaligen gemischt-ganzzahligen nichtlinearen Optimierungsproblemen (MINLP) können PWL-Funktionen Nichtlinearitäten approximieren, wodurch ein MILP-Modell entsteht – das verbessert die Rechenbarkeit erheblich.

Ziele des Projekts sind (1) die Weiterentwicklung des Stands der Technik für die Modellierung und Anwendung von PWL-Funktionen, (2) die Entwicklung passender Lösungsverfahren sowie (3) die Vorstellung eines nicht-konvexen geschachtelten Benders-Dekompositionsverfahrens für MINLP-Probleme auf Basis von PWL-Approximationen.

Zum ersten Ziel: Zwei bestehende MILP-Modelle zur PWL-Approximation werden verglichen, wobei sich ein Modell theoretisch und praktisch über diverse Benchmarks hinweg als überlegen zeigt. Es wird ein allgemeiner Rahmen für Probleme unter Einsatz von PWL-Funktionen – darunter Regression, Clustering und Merkmalsauswahl – vorgeschlagen und auf Support Vector Machines ausgeweitet. Zudem wird ein Modell aus der Literatur analysiert und um drei Verbesserungen ergänzt.

Zum zweiten Ziel: Eine kombinatorische Benders-Dekomposition mit integrierter Ausreißererkennung zur PWL-Berechnung wird vorgestellt und zeigt deutliche Geschwindigkeitsvorteile. Außerdem wird eine effiziente Methode zur Approximation nichtlinearer Funktionen mittels PWL entwickelt, die Beschleunigungen um bis zu 100.000-fach gegenüber dem Stand der Technik erreicht. Soweit bekannt, ist dies aktuell der schnellste Ansatz zur univariaten PWL-Approximation; mehrere Benchmark-Probleme wurden erstmals gelöst. Ein entsprechendes R-Package wurde veröffentlicht. Zusätzlich wird ein spatial Branch-and-Bound-Algorithmus entwickelt, um Optimierungsprobleme mit univariaten PWL-Funktionen effizient zu lösen.

Zum dritten Ziel: Ein mehrstufiges Verfahren, die sogenannte nicht-konvexe geschachtelte Benders-Dekomposition, wird eingeführt, mit dem mehrstufige (stochastische) MINLP-Probleme exakt lösbar sind. Dabei werden PWL-Funktionen genutzt, um das MINLP als MILP zu approximieren, wobei diese Darstellung bei Bedarf dynamisch verfeinert wird. Die entstehenden MILPs werden zerlegt und bestimmte Lagrange-Dualprobleme gelöst, um nicht-konvexe Näherungen der Wertfunktionen zu erzeugen. Da dieser Schritt rechnerisch anspruchsvoll ist, werden alternative Strategien geprüft, die effizienter sind oder Schnitte mit günstigeren Eigenschaften (z.B. Pareto-Optimalität) liefern. Zur Vertiefung des theoretischen Verständnisses werden die mathematischen Eigenschaften und Zusammenhänge der genutzten Konzepte – wie Lipschitz-Regularisierung und Schnittgenerierung – untersucht und tiefere Verbindungen offengelegt.

3 Progress Report

This project aims at contributing to the body of knowledge in the area of piecewise linear (PWL) functions. A continuous PWL function consists out of a finite number of affine functions which are connected at breakpoints. Such PWL functions can be modeled exactly with mixed-integer linear programming (MILP) methods, which makes them particularly suitable for incorporation into MILP models.

Two streams of own previous research are important for this research project, among others: (1) The computation of PWL functions [13, 24, 25, 26, 27] and (2) (multi-stage) linear and non-convex Benders decomposition [3, 17, 22, 23].

The research associated with this project can be grouped broadly into three main areas, which we discuss in detail in the following. **The associated project results are highlighted in bold face.**

3.1 Enhancement of MILP Models for and with Univariate Functions

Warwicker and Rebennack present a comparison between two MILP formulations for PWL functions, [15] and [27], which were published in the same journal at similar times. The authors present a theoretical and experimental comparison on a number of benchmark instances from both papers, to allow readers to take an objective view as to which model is preferable. Although neither formulation is preferable across all tested data sets, the formulation from [27] is faster overall, and contains fewer complicating binary variables and sparse constraints. Validity of the results are ensured through carefully theoretical analysis and computational tests on known problem instances. For the details, please refer to the associated publication **[30]**.

Applications of PWL function fitting to machine learning problems, including regression and clustering, are presented by Warwicker and Rebennack. Models for clusterwise regression [19] and PWL regression [27] are presented alongside novel models for ordered clusterwise regression and clusterwise PWL regression. Since these four models share structural similarities, they are positioned within a single framework whereby tailored solution algorithms, inbuilt outlier detection and feature selection, can be applied. A combinatorial Benders decomposition is developed which shows great computational performance with speedups of up to two orders of magnitude. The validity of the results are demonstrated through benchmarking on five different data sets from the literature, including the debris flow data [16, 18]. For the technical details, please refer to the associated publication **[33]**. John Warwicker presented this research at the INFORMS Annual Meeting 2021 and at the 2023 "German SIAM Student Chapters Meet Algorithmic Optimization" workshop.

Warwicker and Rebennack extend the previous framework to models for support vector machines using PWL functions. Various formulations are presented for different settings of piecewise linearity and connectedness of the model, as well as the incorporation of feature selection. The paper presents three different MILP models, incorporating PWL functions. These models are theoretically compared and computational results are presented. Validity is ensured through careful model development and computational tests on different data sets. For the technical details, please refer to the associated publication **[35]**.

The current state-of-the-art in PWL function modeling and the efficient computation of PWL functions are summarized as encyclopedia articles, authored by Warwicker and Rebennack. The first article presents the following six different MILP formulations for PWL function representation: SOS2, Convex, Disaggregated, Incremental, Multiple-choice and Logarithmic. The second article provides are overview about continuous PWL regression for univariate functions. Please refer to the two publications for the details **[32]** and **[34]**.

Finally, Goldberg et al. improve upon a previous MINLP formulation for PWL function fitting presented previously in [9]. The previous model includes a fault whereby the outputted PWL function may not be injective. Three alternative formulations are proposed to overcome this limitation, and constraints that enforce an ordering on the data points to segments, such as those presented in [27], are the most effective. For the technical details, please refer to the associated publication **[10]**.

3.2 Tailored Solution Methods for MILP Models for and with Univariate Functions

An efficient solution approach to the MILP model presented in [27] has been developed by Warwicker and Rebennack. After implementing inbuilt outlier detection, the authors tailor a combinatorial Benders decomposition [2] to the model. Since the model contains many binary variables, has a symmetric structure and utilizes big-$\mathcal{M}$ constructs, this approach is particularly well-suited. Alongside initialization settings, cut generation and a tailored branching approach, the authors find speedups of up to 12,000 times compared to the monolithic model. For the technical details, we refer to the paper **[31]**. John Warwicker presented this research at OR 2022. The paper was a "research highlight" at the July issue in 2023 of the ISE magazine.

A novel approach for approximating non-linear continuous functions through PWL functions is presented by Warwicker and Rebennack. By incorporating the linear time algorithm for PWL regression presented by [11] within the framework proposed by [27], a method that is up to five orders of magnitude faster than the state-of-the-art is obtained which is able so solve six benchmark functions for the first time. For the technical details, please refer to the paper **[36]**.

The developed algorithms for the very efficient computation of univariate PWL functions, approximating given data points or a continuous function, have also been implemented and made publicly available in the R-package `pwlapprox2d` **[29]**.

Hübner, Gupte and Rebennack develop a spatial branch-and-bound method for solving optimization problems containing (possibly discontinuous) PWL functions. The core novel idea is to provide fast and tight updates of the convex envelope after a spatial branching step. This turns out to be theoretically and computationally superior than standard branch-and-cut approaches from the literature. Computational tests are performed on knapsack and network flow problems with comparisons to the latest logarithmic-sized models from the literature. The results show speedups of more than three order of magnitude for large problem instances compared to the PWL solver in Gurobi and demonstrate the validity of the results through intensive benchmarking. The code and all data necessary to replicate the results are publicly available in a permanent online repository. For the technical details, please refer to the associated paper **[12]**. This paper was selected as "runner up" for the ICS Student Paper Award of the INFORMS Computing Society in 2025.

3.3 Application in Non-convex Nested Benders Decomposition

Benders decomposition has been originally proposed by Benders in 1962 for linear optimization problems with special structure [1]. Benders proposed to decompose the problem into a master problem and a subproblem, where the subproblem information is communicated to the master problem via a value function, which is itself iteratively refined through *linear cuts*. Benders decomposition has been extended in various ways, *e.g.*, to solve multi-stage linear optimization problems, to incorporate non-convexity (both due to integer variables and non-convex functions) in the master problem and convex non-linearity in the subproblem [8, 20, 21, 28]. There has also been extensions to integer linear subproblems with the Stochastic Dual Dynamic integer Programming (SDDiP) algorithm, with the restriction to binary state-variables [37]. All these extensions have in common that the value functions remain convex and can thus be approximated by *linear cuts* [5].

Füllner and Rebennack present the first Benders decomposition-based algorithm which can also handle general non-convexities in the subproblems, without any state-space restrictions.

They coin this algorithm “non-convex nested Benders decomposition”. As the resulting value functions are non-convex, *linear cuts* are no longer sufficient. Therefore, the authors introduce the *lift-and-project cuts*. These cuts are linear in a lifted space and become *non-convex* after projection into the original space, yielding valid and tight *non-convex cuts* for the original value functions, which ensures finite convergence of the proposed decomposition algorithm. Therefore, the algorithm first approximates the original non-convex subproblem as a mixed-integer linear subproblem via PWL functions, generates a linear cut and projects it back to the original subproblem. This step also requires a dynamic refinement of the binary approximation of the state-space, in order to apply SDDiP like methods as a subroutine. Validity of the results are ensured through rigorous mathematical proofs and through computational tests on the unit commitment problem – an important problem in power systems optimization [14]. For the technical details, please refer to the associated paper **[4]**. Christian Füllner presented this research at EURO 2021 and at OR 2021.

A theoretical analysis of the proposed nonconvex nested Benders decomposition algorithm is performed by Füllner and Rebennack. They extend their own work by (1) considering also stochastic multi-stage non-convex optimization problems and by (2) carefully engaging a regularization ensuring Lipschitz continuity of the lift-and-project (non-linear) cuts. This eliminates the need for one of the assumptions made in the original nonconvex nested Benders decomposition algorithm. As a by product, new insights on the role of the so-called copy-constraints – they are key in SDDiP – are gained. Validity of the results are ensured through rigorous mathematical proofs and instructive examples. For the details, please refer to the paper **[7]**.

The work by Füllner, Sun and Rebennack generalizes the so-called Lagrangian cuts, present in SDDiP-type algorithms. A general and flexible new framework for generating a broad class of Lagrangian cuts is presented which allows the computation of Lagrangian cuts with different favorable properties, such as maximal depth, being facet-defining or Pareto-optimal. Computational results for capacitated lot-sizing and capacitated facility location problems demonstrate that the lower bounds in SDDiP can be improved significantly with these new type of Lagrangian cuts. Validity of the results are ensured through rigorous mathematical proofs and extensive computational benchmarking on known problem instances. For the technical details, please refer to the associated paper **[6]**. Christian Füllner presented this research at ECSO-CMS 2022 and at OR 2022.

References

[1] J. F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, pages 238–252, 1962.

[2] G. Codato and M. Fischetti. Combinatorial benders' cuts for mixed-integer linear programming. *Operations Research*, 54(4):756–766, 2006.

[3] S. M. Frank and S. Rebennack. Optimal design of mixed ac–dc distribution systems for commercial buildings: A nonconvex generalized benders decomposition approach. *European Journal of Operational Research*, 242(3):710–729, 2015.

[4] C. Füllner and S. Rebennack. Non-convex nested Benders decomposition. *Mathematical Programming*, 196(1):987–1024, 2022.

[5] C. Füllner and S. Rebennack. Stochastic dual dynamic programming and its variants—a review. *SIAM Review*, 67(3):415–539, 2025.

[6] C. Füllner, X. A. Sun, and S. Rebennack. A new framework to generate Lagrangian cuts in multistage stochastic mixed-integer programming. *https://optimization-online.org/2024/08/a-new-framework-to-generate-lagrangian-cuts-in-multistage-stochastic-mixed-integer-programming/*, 2024.

[7] C. Füllner, X. A. Sun, and S. Rebennack. On Lipschitz regularization and Lagrangian cuts in multistage stochastic mixed-integer linear programming. *https://optimization-online.org/2024/08/on-lipschitz-regularization-and-lagrangian-cuts-in-multistage-stochastic-mixed-integer-linear-programming/*, 2024.

[8] A. M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972.

[9] N. Goldberg, Y. Kim, S. Leyffer, and T. D. Veselka. Adaptively refined dynamic program for linear spline regression. *Computational Optimization and Applications*, 58:523–541, 2014.

[10] N. Goldberg, S. Rebennack, Y. Kim, V. Krasko, and S. Leyffer. MINLP formulations for continuous piecewise linear function fitting. *Computational Optimization and Applications*, 79:223–233, 2021.

[11] S. L. Hakimi and E. F. Schmeichel. Fitting polygonal functions to a set of points in the plane. *CVGIP: Graphical Models and Image Processing*, 53(2):132–136, 1991.

[12] T. Hübner, A. Gupte, and S. Rebennack. Spatial branch-and-bound for nonconvex separable piecewise linear optimization. *INFORMS Journal on Computing*, 2025.

[13] J. Kallrath and S. Rebennack. *Computing Area-Tight Piecewise Linear Overestimators, Underestimators and Tubes for Univariate Functions*, pages 273–292. Springer New York, 2014.

[14] A. Kaya, A. J. Conejo, and S. Rebennack. Fifty years of power systems optimization. *European Journal of Operational Research*, 2025.

[15] L. Kong and C. T. Maravelias. On the derivation of continuous piecewise linear approximating functions. *INFORMS Journal on Computing*, 32(3):531–546, 2020.

[16] V. Krasko and S. Rebennack. Two-stage stochastic mixed-integer nonlinear programming model for post-wildfire debris flow hazard management: Mitigation and emergency evacuation. *European Journal of Operational Research*, 263(1):265–282, 2017.

[17] T. Lohmann and S. Rebennack. Tailored benders decomposition for a long-term power expansion model with short-term demand response. *Management Science*, 63(6):2027–2048, 2017.

[18] K. McCoy, V. Krasko, P. Santi, D. Kaffine, and S. Rebennack. Minimizing economic impacts from post-fire debris flows in the western united states. *Natural Hazards*, 83(1):149–176, 2016.

[19] Y. W. Park, Y. Jiang, D. Klabjan, and L. Williams. Algorithms for generalized clusterwise linear regression. *INFORMS Journal on Computing*, 29(2):301–317, 2017.

[20] M. V. Pereira and L. M. Pinto. Multi-stage stochastic optimization applied to energy planning. *Mathematical programming*, 52(1):359–375, 1991.

[21] R. Rahmaniani, T. G. Crainic, M. Gendreau, and W. Rei. The Benders decomposition algorithm: A literature review. *European Journal of Operational Research*, 259(3):801–817, 2017.

[22] S. Rebennack. Generation expansion planning under uncertainty with emissions quotas. *Electric Power Systems Research*, 114:78–85, 2014.

[23] S. Rebennack. Combining sampling-based and scenario-based nested Benders decomposition methods: application to stochastic dual dynamic programming. *Mathematical Programming*, 156(1):343–389, 2016.

[24] S. Rebennack. Computing tight bounds via piecewise linear functions through the example of circle cutting problems. *Mathematical Methods of Operations Research*, 84(1):3–57, 2016.

[25] S. Rebennack and J. Kallrath. Continuous piecewise linear delta-approximations for bivariate and multivariate functions. *Journal of Optimization Theory and Applications*, 167(1):102–117, 2015.

[26] S. Rebennack and J. Kallrath. Continuous piecewise linear delta-approximations for univariate functions: computing minimal breakpoint systems. *Journal of Optimization Theory and Applications*, 167(2):617–643, 2015.

[27] S. Rebennack and V. Krasko. Piecewise linear function fitting via mixed-integer linear programming. *INFORMS Journal on Computing*, 32(2):507–530, 2020.

[28] G. Steeger and S. Rebennack. Dynamic convexification within nested Benders decomposition using lagrangian relaxation: An application to the strategic bidding problem. *European Journal of Operational Research*, 257(2):669–686, 2017.

[29] J. Warwicker and S. Rebennack. pwlapprox2d: Univariate piecewise linear regression for discrete data and continuous functions. R package version 2.7.3, `https://CRAN.R-project.org/package=pwlapprox2d`, 2025.

[30] J. A. Warwicker and S. Rebennack. A comparison of two mixed-integer linear programs for piecewise linear function fitting. *INFORMS Journal on Computing*, 34(2):1042–1047, 2022.

[31] J. A. Warwicker and S. Rebennack. Generating optimal robust continuous piecewise linear regression with outliers through combinatorial Benders decomposition. *IISE Transactions*, 55(8):755–767, 2023.

[32] J. A. Warwicker and S. Rebennack. *Mixed-Integer Programming Formulations for Piecewise Linear Functions*, pages 1–7. Springer, 2023.

[33] J. A. Warwicker and S. Rebennack. A unified framework for bivariate clustering and regression problems via mixed-integer linear programming. *Discrete Applied Mathematics*, 336:15–36, 2023.

[34] J. A. Warwicker and S. Rebennack. *Univariate Continuous Piecewise Linear Regression*, pages 1–8. Springer, 2023.

[35] J. A. Warwicker and S. Rebennack. Support vector machines within a bivariate mixed-integer linear programming framework. *Expert Systems With Applications*, 245:122998, 2024.

[36] J. A. Warwicker and S. Rebennack. Efficient continuous piecewise linear regression for linearising univariate non-linear functions. *IISE Transactions*, 57(3):231–245, 2025.

[37] J. Zou, S. Ahmed, and X. A. Sun. Stochastic dual dynamic integer programming. *Mathematical Programming*, 175(1):461–502, 2019.

4 Published Project Results

4.1 Publications with scientific quality assurance

Peer-Reviewed Journals:

1. Füllner, C. and Rebennack, S. (2022). *Non-convex nested Benders decomposition.* **Mathematical Programming**, 196, 987–1024, doi: 10.1007/s10107-021-01740-0 (Open Access)

2. Goldberg, N., Rebennack, S., Kim, Y., Krasko, V. and Leyffer, S. (2021). *MINLP formulations for continuous piecewise linear function fitting.* **Computational Optimization and Applications**, 79: 223-233, doi: 10.1007/s10589-021-00268-5 (Open Access)

3. Hübner, T., Gupte, A. and Rebennack, S. (2025). *Spatial branch-and-bound for nonconvex separable piecewise linear optimization*. **INFORMS Journal on Computing**, Article in Advance. doi: 10.1287/ijoc.2024.0755 (Open Access)
   → Paper awarded as runner up at ICS Student Paper Prize, INFORMS
4. Warwicker, J. A. and Rebennack, S. (2022). *A Comparison of two mixed-integer linear programs for piecewise linear function fitting.* **INFORMS Journal on Computing**, 34(2):1042-1047, doi: 10.1287/ijoc.2021.1114 (Open Access)
5. Warwicker, J. A. and Rebennack, S. (2023). *Generating optimal robust continuous piecewise linear regression with outliers through combinatorial Benders decomposition.* **IISE Transactions**, 55(8), 755–767, doi: 10.1080/24725854.2022.2107249 (Open Access)
   → Paper is a "research highlight" in the July 2023 issue of the ISE magazine
6. Warwicker, J. A. and Rebennack, S. (2023). *A unified framework for bivariate clustering and regression problems via mixed-integer linear programming.* **Discrete Applied Mathematics**, 336, 15-36, doi: 10.1016/j.dam.2023.03.010 (Open Access)
7. Warwicker, J. A. and Rebennack, S. (2024) *Support vector machines within a bivariate mixed-integer linear programming framework.* **Expert Systems with Applications**, 245,122998, doi: 10.1016/j.eswa.2023.122998 (Open Access)
8. Warwicker, J. A. and Rebennack, S. (2025). *Efficient continuous piecewise linear regression for linearising univariate non-linear functions.* **IISE Transactions**, 57(3), 231–245, doi: doi: 10.1080/24725854.2023.2299809 (Open Access)

Peer-reviewed book chapters:

9. Warwicker, J. A. and Rebennack, S. (2023). *Mixed-integer programming formulations for piecewise linear functions.* In: Pardalos, P. M., Prokopyev, O. A. (eds) Encyclopedia of Optimization. Springer, Cham, doi: 10.1007/978-3-030-54621-2_791-1
10. Warwicker, J. A. and Rebennack, S. (2023). *Univariate Continuous Piecewise Linear Regression.* In: Pardalos, P. M., Prokopyev, O. A. (eds) Encyclopedia of Optimization. Springer, Cham, doi: 10.1007/978-3-030-54621-2_727-1

4.2 Other publications and published results

11. Füllner, C., Sun, X. A. and Rebennack, S. (2024). *A new framework to generate Lagrangian cuts in multistage stochastic mixed-integer programming*.
12. Füllner, C., Sun, X. A. and Rebennack, S. (2024). *On Lipschitz regularization and Lagrangian cuts in multistage stochastic mixed-integer linear programming*.
13. Warwicker, J.A. and Rebennack, S. (2025). pwlapprox2d: Univariate piecewise linear regression for discrete data and continuous functions. R package. https://CRAN.R-project.org/package=pwlapprox2d