

Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint

ISSN 2198-5855

**Optimal distributed control of a nonlocal convective
Cahn-Hilliard equation by the velocity in 3D**

Elisabetta Rocca^{1, 2}, Jürgen Sprekels^{2, 3}

submitted: April 10, 2014

¹ Università di Milano
Dipartimento di Matematica
Via Saldini 50
20133 Milano
Italy
E-Mail elisabetta.rocce@unimi.it

² Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: juergen.sprekels@wias-berlin.de
E-Mail: elisabetta.rocce@wias-berlin.de

³ Humboldt-Universität zu Berlin
Institut für Mathematik
Rudower Chaussee 25
12489 Berlin
Germany

No. 1942
Berlin 2014



2010 *Mathematics Subject Classification.* 49J20, 49J50, 35R09, 45K05, 74N99.

Key words and phrases. Distributed optimal control, first-order necessary optimality conditions, nonlocal models, integrodifferential equations, convective Cahn–Hilliard equation, phase separation.

The work of E.R. was supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase) and by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: <http://www.wias-berlin.de/>

Abstract

In this paper we study a distributed optimal control problem for a nonlocal convective Cahn–Hilliard equation with degenerate mobility and singular potential in three dimensions of space. While the cost functional is of standard tracking type, the control problem under investigation cannot easily be treated via standard techniques for two reasons: the state system is a highly nonlinear system of PDEs containing singular and degenerating terms, and the control variable, which is given by the velocity of the motion occurring in the convective term, is nonlinearly coupled to the state variable. The latter fact makes it necessary to state rather special regularity assumptions for the admissible controls, which, while looking a bit nonstandard, are however quite natural in the corresponding analytical framework. In fact, they are indispensable prerequisites to guarantee the well-posedness of the associated state system. In this contribution, we employ recently proved existence, uniqueness and regularity results for the solution to the associated state system in order to establish the existence of optimal controls and appropriate first-order necessary optimality conditions for the optimal control problem.

1 Introduction

This paper is concerned with the study of a distributed control problem for a Cahn–Hilliard type PDE system. More precisely, we investigate the case of a *nonlocal* Cahn–Hilliard equation with convective term, degenerate mobility and singular potential. In fact, while the standard Cahn–Hilliard equation (cf., e.g., [3, 4, 5]) is widely used, it seems that a more realistic version of the Cahn–Hilliard equation can be characterized by a (spatially) nonlocal free energy. Although the physical relevance of nonlocal interactions was already pointed out in the pioneering paper [29] (see also [12, 4.2] and references therein), the isothermal and nonisothermal models containing nonlocal terms have only recently been studied from the analytical viewpoint (cf., e.g., [1, 9, 15, 17, 18, 22] and the references given there).

The main difference between local and nonlocal models is given by the choice of the interaction potential. Typically, the nonlocal contribution to the free energy has the form $\int_{\Omega} k(x, y) |\varphi(x) - \varphi(y)|^2 dy$, with a given symmetric kernel k defined on $\Omega \times \Omega$, where Ω denotes a (sufficiently regular and bounded) domain in \mathbb{R}^3 in which the phase separation takes place; its local Ginzburg–Landau counterpart is given by $(\sigma/2)|\nabla\varphi(x)|^2$, where the positive parameter σ is a measure for the thickness of the interface. Here, φ represents the local concentration of one of the two phases, which typically attains values in a bounded interval, say, in $[0, 1]$. The local potential can be obtained as a formal limit as $m \rightarrow \infty$ from the nonlocal one with the choice $k(x, y) = m^5 k(|m(x - y)|^2)$, where k is a nonnegative function with support in $[0, 1]$. This

follows from the formula (which was formally deduced in [21])

$$\int_{\Omega} m^5 k(|m(x-y)|^2) |\varphi(x) - \varphi(y)|^2 dy = \int_{\Omega_m(x)} k(|z|^2) \left| \frac{\varphi\left(x + \frac{z}{m}\right) - \varphi(x)}{\frac{1}{m}} \right|^2 dz$$

$$\xrightarrow{m \rightarrow \infty} \int_{\mathbb{R}^3} k(|z|^2) \langle \nabla \varphi(x), z \rangle^2 dz = \frac{\sigma}{2} |\nabla \varphi(x)|^2,$$

for a sufficiently regular φ , where $\sigma = 2/3 \int_{\mathbb{R}^3} k(|z|^2) |z|^2 dz$ and $\Omega_m(x) = m(\Omega - x)$. Here we have used that $\int_{\mathbb{R}^3} k(|z|^2) \langle e, z \rangle^2 dz = 1/3 \int_{\mathbb{R}^3} k(|z|^2) |z|^2 dz$ for every unit vector $e \in \mathbb{R}^3$. As a consequence, the local Cahn–Hilliard equation can be viewed as an approximation of the nonlocal one.

In the seminal paper [11], the authors established the existence of a weak solution to the local Cahn–Hilliard equation with degenerate mobility and singular potentials endowed with no-flux boundary conditions. However, in the local case no uniqueness proof is known in case of degenerate mobility and singular potential. This is one of the main advantages of considering the nonlocal potential: for the nonlocal Cahn–Hilliard system, indeed, in the case of periodic boundary conditions, an existence and uniqueness result was proved in [18]. Later, a more general case was considered in [15]. More recently, the convergence to single equilibria was studied in [24, 25] (cf. also [16] for further results), and in [14] the existence of a global attractor for a convective nonlocal Cahn–Hilliard equation with degenerate mobility and singular potential was proved in the three-dimensional case. Moreover, for the two-dimensional case also the long-time dynamics of its coupling with the Navier–Stokes equation (the nonlocal version of the so-called H-model) was analyzed in [14]. For this model uniqueness of weak solutions and existence of the global attractor in two dimensions has been recently proved in [13].

Concerning the problem of deriving first-order necessary optimality conditions for optimal control problems involving *local* Cahn–Hilliard equations, we can quote the following references: in [32], the authors studied the case of a polynomially growing potential f (in (1.3)) with constant mobility m in (1.2), while only more recently in [19] and [20], the case of the double obstacle potential $f = I_{[0,1]}$ in (1.3) with constant mobility m in (1.2) was investigated; first-order necessary optimality conditions were obtained by means of a regularization procedure. Moreover, the convective 1D case has been dealt with in the very recent contribution [33]. Let us finally recall the papers [8] and [7], where the authors study the optimal control problem associated with a non-standard phase-field model of Cahn–Hilliard type, and [2], respectively, where optimization techniques were used in order to solve variational inequalities related to Allen–Cahn and Cahn–Hilliard equations.

While optimal control problems for certain classes of PDEs coupled with nonlocal *boundary conditions* have already been studied in the literature (cf., e.g., [10, 26, 27, 28]), to our best knowledge no analytical contribution exists in the literature to the study of optimal control problems for nonlocal phase-field models of convective Cahn–Hilliard type and, more generally, for nonlocal PDEs where the nonlocal operator appears in the PDEs and not on the boundary.

Throughout this paper, we will generally assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and outward unit normal \mathbf{n} , and we denote $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$, where $T > 0$ is a prescribed final time. We then consider the following control problem:

(CP) Minimize the cost functional

$$J(\varphi, \mathbf{v}) = \frac{\beta_1}{2} \int_0^T \int_{\Omega} |\varphi - \varphi_Q|^2 dx dt + \frac{\beta_2}{2} \int_{\Omega} |\varphi(T) - \varphi_{\Omega}|^2 dx + \frac{\beta_3}{2} \int_0^T \int_{\Omega} |\mathbf{v}|^2 dx dt, \quad (1.1)$$

subject to the initial-boundary value problem (the *state system*)

$$\varphi_t - \operatorname{div}(m(\varphi)\nabla\mu) = -\mathbf{v} \cdot \nabla\varphi \quad \text{in } Q, \quad (1.2)$$

$$\mu = f'(\varphi) + w \quad \text{in } Q, \quad (1.3)$$

$$w(x, t) = \int_{\Omega} k(|x - y|)(1 - 2\varphi(y, t)) dy \quad \text{in } Q, \quad (1.4)$$

$$m(\varphi)\nabla\mu \cdot \mathbf{n} = 0 \quad \text{on } \Sigma, \quad (1.5)$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega, \quad (1.6)$$

and to the constraint that the velocity \mathbf{v} , which plays the role of the control, belongs to a suitable closed, bounded and convex subset (to be specified later) of the space

$$\mathcal{V} := \{\mathbf{v} \in L^2(0, T; H_{div}^1(\Omega)) \cap L^\infty(Q)^3 : \exists \mathbf{v}_t \in L^2(0, T; L^3(\Omega)^3)\}, \quad (1.7)$$

where

$$H_{div}^1(\Omega) := \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div}(\mathbf{v}) = 0\}. \quad (1.8)$$

Notice that the velocity is assumed divergence free, and observe that through the convective term $-\mathbf{v} \cdot \nabla\varphi$ the coupling between control and state is nonlinear. This nonlinear coupling between control and state is the reason for the strong and a bit nonstandard regularity assumption for the time derivative of the control \mathbf{v} . We also remark that both $H_{div}^1(\Omega)$ and \mathcal{V} are Banach spaces when equipped with their natural norms.

The singular potential f will be taken in the typical logarithmic form (cf. the original paper [4])

$$f(\varphi) = \varphi \log(\varphi) + (1 - \varphi) \log(1 - \varphi),$$

and the mobility m , which degenerates at the pure phases $\varphi = 0$ and $\varphi = 1$, has to satisfy the compatibility condition (cf. [11], [15], [25])

$$m(\varphi) = \frac{c_0}{f''(\varphi)} = c_0\varphi(1 - \varphi) \quad \text{with some constant } c_0 > 0,$$

which entails that we have the relations

$$m(\varphi)f''(\varphi) \equiv c_0, \quad m(\varphi)\nabla\mu = c_0\nabla\varphi + m(\varphi)\nabla w. \quad (1.9)$$

Moreover, throughout this paper we assume that the given constants $\beta_1, \beta_2, \beta_3$ in (1.1) are nonnegative, while $\varphi_Q \in L^2(Q)$ and $\varphi_{\Omega} \in L^2(\Omega)$ represent prescribed target functions of the cost functional J . We could generalize both the expressions of J and of the potential f ,

but we restrict ourselves to the above situation for the sake of a simpler exposition. In particular, we could consider the case when

$$f \in C^2(0, 1) \text{ is strictly convex in } (0, 1), \quad \text{Im}(f')^{-1} = [0, 1],$$

$$\frac{1}{f''} \text{ is strictly concave in } (0, 1),$$

and, for example,

$$m \in C^2([0, 1]) \quad \text{satisfies} \quad m(\varphi)f''(\varphi) \geq c_0 > 0 \text{ for every } \varphi \in [0, 1].$$

Other interesting problems would be related to the case of more general potentials and mobilities, but also to the optimal control problem related to the coupling of (1.2)–(1.6) with a Navier–Stokes system governing the evolution of the velocity \mathbf{v} . The existence of weak solutions to such coupled systems and their long-time behavior have recently been studied in [14] in the two- and three-dimensional cases. The analysis of an associated control problem in the 2D case will be the subject of a forthcoming paper.

Plan of the paper. The paper is organized as follows: in Section 2, we recall known results regarding the well-posedness of the PDE system (1.2)–(1.6) as well as the related separation property. We also prove a continuous dependence result (Lemma 2.1) which is needed for the analysis of the control problem. In Section 3, we prove the main results of this paper concerning existence and first-order necessary optimality conditions for the optimal control problem **(CP)**.

Throughout this paper we will denote the norm of a Banach space E by $\|\cdot\|_E$. In the following, we will make repeated use of Young's inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a, b \in \mathbb{R} \quad \text{and } \delta > 0, \quad (1.10)$$

as well as of the fact that for three dimensions of space the embeddings $H^1(\Omega) \subset L^p(\Omega)$, $1 \leq p \leq 6$, and $H^2(\Omega) \subset C^0(\overline{\Omega})$ are continuous and (in the first case only for $1 \leq p < 6$) compact. Moreover, we recall that for smooth and bounded three-dimensional domains there hold the special Gagliardo–Nirenberg inequalities

$$\|v\|_{L^3(\Omega)} \leq \widehat{K}_1 \left(\|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^1(\Omega)}^{1/2} + \|v\|_{L^2(\Omega)} \right) \quad \forall v \in H^1(\Omega), \quad (1.11)$$

$$\|v\|_{L^4(\Omega)} \leq \widehat{K}_2 \left(\|v\|_{L^2(\Omega)}^{1/4} \|v\|_{H^1(\Omega)}^{3/4} + \|v\|_{L^2(\Omega)} \right) \quad \forall v \in H^1(\Omega), \quad (1.12)$$

where the constants $\widehat{K}_1 > 0$ and $\widehat{K}_2 > 0$ depend only on Ω ; observe that (1.10) and the continuity of the embedding $W^{1,4}(\Omega) \subset L^\infty(\Omega)$ imply that for every $\delta > 0$ it holds

$$\|v\|_{L^3(\Omega)}^2 \leq \delta \|v\|_{H^1(\Omega)}^2 + \frac{\widehat{K}_3}{\delta} \|v\|_{L^2(\Omega)}^2 \quad \forall v \in H^1(\Omega), \quad (1.13)$$

$$\|v\|_{L^\infty(\Omega)}^2 \leq \delta \|v\|_{H^2(\Omega)}^2 + \frac{\widehat{K}_4}{\delta} \|v\|_{H^1(\Omega)}^2 \quad \forall v \in H^2(\Omega), \quad (1.14)$$

where also $\widehat{K}_3 > 0$ and $\widehat{K}_4 > 0$ depend only on Ω . Finally, for the sake of a shorter exposition, we denote by \mathcal{K} the integral operator that assigns to φ the function w through (1.4); that is, we put

$$\mathcal{K}(\varphi)(x, t) := \int_{\Omega} k(|x - y|)(1 - 2\varphi(y, t)) \, dy. \quad (1.15)$$

2 Well-posedness of the state system

In the following, we study the state system (1.2)–(1.6). To fix things, we assume for the set of admissible controls:

$$\text{(H1)} \quad \mathcal{V}_{\text{ad}} := \left\{ \mathbf{v} = (v_1, v_2, v_3) \in \mathcal{V} : \begin{aligned} &\tilde{v}_{1_i} \leq v_i \leq \tilde{v}_{2_i} \text{ a.e. in } Q, \quad i = 1, 2, 3, \\ &\|\mathbf{v}\|_{L^2(0, T; H^1(\Omega)^3)} + \|\mathbf{v}_t\|_{L^2(0, T; L^3(\Omega)^3)} \leq V \end{aligned} \right\},$$

where $V > 0$ is a given constant and $\tilde{v}_{1_i}, \tilde{v}_{2_i} \in L^\infty(Q)$, $i = 1, 2, 3$, are given threshold functions; we generally assume that $\mathcal{V}_{\text{ad}} \neq \emptyset$.

Observe that \mathcal{V}_{ad} is a bounded, closed, and convex subset of \mathcal{V} , which is certainly contained in some bounded open subset of \mathcal{V} . For convenience, we fix such a set once and for all, noting that any other such set could be used instead:

$$\text{(H2)} \quad \mathcal{V}_R \subset \mathcal{V} \text{ is an open set satisfying } \mathcal{V}_{\text{ad}} \subset \mathcal{V}_R \text{ such that, for all } \mathbf{v} \in \mathcal{V}_R,$$

$$\|\mathbf{v}\|_{L^2(0, T; H^1(\Omega)^3)} + \|\mathbf{v}\|_{L^\infty(Q)^3} + \|\mathbf{v}_t\|_{L^2(0, T; L^3(\Omega)^3)} \leq R. \quad (2.1)$$

Before stating some known results on the well-posedness of the state system (1.2)–(1.6), we now formulate the general assumptions for the problem data. We remark at this place that not all of these assumptions are needed to ensure the respective results concerning existence, separation, uniqueness, and regularity; however, they are indispensable prerequisites for the continuous dependence result of Lemma 2.1 below, which will be needed for the derivation of necessary optimality conditions for the control problem. Since we focus on optimal control here, we have decided to impose the corresponding (stronger) conditions from the very beginning in order to avoid any confusion. We make the following assumptions:

$$\text{(H3)} \quad \varphi_0 \in H^2(\Omega), \quad \text{and there is some } \kappa_0 > 0 \text{ such that } 0 < \kappa_0 \leq \varphi_0 \leq 1 - \kappa_0 < 1 \text{ a.e. in } \Omega.$$

$$\text{(H4)} \quad f(\varphi) = \varphi \log(\varphi) + (1 - \varphi) \log(1 - \varphi) \quad \text{for } 0 < \varphi < 1, \quad f(0) = f(1) = 0, \\ f(\varphi_0) \in L^1(\Omega).$$

$$\text{(H5)} \quad m(\varphi) = \frac{c_0}{f''(\varphi)} \quad \text{for } 0 < \varphi < 1, \text{ with some } c_0 > 0.$$

$$\text{(H6)} \quad \int_{\Omega} \int_{\Omega} k(|x - y|) \, dx \, dy =: k_0 < +\infty, \quad \sup_{x \in \Omega} \int_{\Omega} |k(|x - y|)| \, dy =: \bar{k} < +\infty.$$

(H7) $\forall p \in [1, +\infty] \exists k_p > 0 : \left\| -2 \int_{\Omega} k(|x-y|) z(y) dy \right\|_{W^{1,p}(\Omega)} \leq k_p \|z\|_{L^p(\Omega)}$
for all $z \in W^{1,p}(\Omega)$.

(H8) For $p \in \{2, 3\}$ there is some $s_p > 0$ such that for all $z \in W^{1,p}(\Omega)$ it holds
 $\left\| -2 \int_{\Omega} k(|x-y|) z(y) dy \right\|_{W^{2,p}(\Omega)} \leq s_p \|z\|_{W^{1,p}(\Omega)}$.

We now collect some known results for the state system. At first, we claim that the system (1.2)–(1.6) admits under the hypotheses **(H1)–(H8)** for any $\mathbf{v} \in \mathcal{V}_R$ a unique solution triple (φ, w, μ) such that

$$\varphi \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega)) \cap C^0(\overline{Q}). \quad (2.2)$$

Indeed, adapting the proof by Gajewski and Zacharias (see [15, Thm. 3.5] and also [14, Thm. 4 and Prop. 4]) to the case $\mathbf{v} \neq 0$, one can first establish the existence of a unique weak solution (φ, w, μ) to (1.2)–(1.6) such that

$$\varphi \in L^2(0, T; H^1(\Omega)), \quad \varphi_t \in L^2(0, T; H^1(\Omega)^*), \quad 0 < \varphi < 1 \quad \text{a.e. in } Q, \quad (2.3)$$

$$\int_{\Omega} \varphi(x, t) dx = \int_{\Omega} \varphi_0(x) dx \quad \forall t \in [0, T], \quad (2.4)$$

$$w \in L^\infty(0, T; W^{1,\infty}(\Omega)). \quad (2.5)$$

Next, we adapt the argument of Londen–Petzeltová (see [25, Prop. 3.1]) to the present situation to conclude that there is a constant $\kappa \in (0, 1)$, which is independent of the choice of $\mathbf{v} \in \mathcal{V}_R$, such that the weak solution satisfies the separation property

$$0 < \kappa \leq \varphi \leq 1 - \kappa < 1 \quad \text{a.e. in } Q. \quad (2.6)$$

Finally, we can argue as in the proof of [25, Thm. 2.2] to conclude the additional regularity property (2.2).

Remark 1. A closer inspection of the abovementioned proofs reveals that there is a constant $\widehat{C}_1 > 0$, which only depends on the data of the system and on the constant R , such that we have

$$\|\varphi\|_{W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap C^0([0,T];H^2(\Omega)) \cap C^0(\overline{Q})} \leq \widehat{C}_1, \quad (2.7)$$

whenever φ is the first component of a solution (φ, w, μ) associated with some $\mathbf{v} \in \mathcal{V}_R$. But then it follows from the hypotheses **(H7)** and **(H8)** that, in particular,

$$\|w_t\|_{L^2(0,T;H^2(\Omega))} + \|w\|_{C^0([0,T];H^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))} \leq \widehat{C}_2, \quad (2.8)$$

where also \widehat{C}_2 only depends on the data and R . Moreover, the separation property (2.6) holds even pointwise for every $(x, t) \in \overline{Q}$, whence it follows that

$$\max_{1 \leq i \leq 4} \|f^{(i)}(\varphi)\|_{C^0(\overline{Q})} \leq \widehat{C}_3, \quad (2.9)$$

where, again, \widehat{C}_3 only depends on the data and R . Therefore, we can conclude from (1.3) that

$$\|\mu_t\|_{L^2(0,T;H^2(\Omega))} + \|\mu\|_{C^0([0,T];H^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))} \leq \widehat{C}_4, \quad (2.10)$$

where also \widehat{C}_4 only depends on the data and R . In the remainder of this paper, we denote $K_1^* := \max_{1 \leq i \leq 4} \widehat{C}_i$.

Remark 2. The separation property (2.6) and the hypotheses **(H4)** and **(H5)** also entail the estimate

$$\frac{c_0}{4} \geq m(\varphi(x, t)) \geq c_0 \kappa(1 - \kappa) > 0 \quad \text{for every } (x, t) \in \overline{Q}. \quad (2.11)$$

This means that under the given hypotheses neither the possible degeneracy of m nor the possible singularity of f' can become active. Also, we may without loss of generality assume (by possibly choosing a larger K_1^*) that

$$\|m(\varphi)\|_{C^0(\overline{Q})} + \|m'(\varphi)\|_{C^0(\overline{Q})} + \|m''(\varphi)\|_{C^0(\overline{Q})} \leq K_1^*. \quad (2.12)$$

We will now show a global stability estimate. We have the following result.

Lemma 2.1. *Let the hypotheses **(H1)–(H8)** be satisfied. Then there exists a constant $K_2^* > 0$, which only depends on the data of the state system and on R , such that it holds: whenever $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_R$ are given and $\varphi_1, \varphi_2 \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega))$ denote the associated solutions to the state system (1.2)–(1.6) and w_1, w_2 the corresponding nonlocal operators according to (1.4), then we have for $\varphi := \varphi_1 - \varphi_2$, $w := w_1 - w_2$, $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$, that*

$$\int_0^t \|\varphi_t(s)\|_{L^2(\Omega)}^2 ds + \|\varphi(t)\|_{H^1(\Omega)}^2 \leq K_2^* \int_0^t \|\mathbf{v}(s)\|_{L^3(\Omega)^3}^2 ds \quad \forall t \in [0, T]. \quad (2.13)$$

PROOF. In the following, the symbol C will denote positive constants, which possibly differ from line to line or even within lines. They may only depend on the problem data and R . To begin with, we observe that (φ, w, μ) satisfies

$$\varphi_t - \operatorname{div}(m(\varphi_1)\nabla\mu_1 - m(\varphi_2)\nabla\mu_2) = -\mathbf{v} \cdot \nabla\varphi_1 - \mathbf{v}_2 \cdot \nabla\varphi \quad \text{a.e. in } Q, \quad (2.14)$$

$$\mu = f'(\varphi_1) - f'(\varphi_2) + w \quad \text{a.e. in } Q, \quad (2.15)$$

$$w(x, t) = -2 \int_{\Omega} k(|x - y|)\varphi(y, t) dy \quad \text{a.e. in } Q, \quad (2.16)$$

$$\varphi(0) = w(0) = \mu(0) = 0 \quad \text{a.e. in } \Omega. \quad (2.17)$$

We also notice that, owing to (1.3), **(H2)** and (1.9), we have

$$\begin{aligned} (m(\varphi_1)\nabla\mu_1 - m(\varphi_2)\nabla\mu_2) &= c_0 \nabla\varphi + (m(\varphi_1)\nabla w_1 - m(\varphi_2)\nabla w_2) \\ &= c_0 \nabla\varphi + (m(\varphi_1) - m(\varphi_2))\nabla w_1 + m(\varphi_2)\nabla w. \end{aligned} \quad (2.18)$$

Hence, testing (2.14) by φ , we have, for every $t > 0$,

$$\begin{aligned}
& \frac{1}{2} \|\varphi(t)\|_{L^2(\Omega)}^2 + c_0 \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds \leq \int_0^t \int_{\Omega} |\mathbf{v}_2| |\varphi| |\nabla \varphi| dx ds \\
& + \int_0^t \int_{\Omega} |m(\varphi_1) - m(\varphi_2)| |\nabla w_1| |\nabla \varphi| dx ds + \int_0^t \int_{\Omega} |m(\varphi_2)| |\nabla w| |\nabla \varphi| dx ds \\
& + \int_0^t \int_{\Omega} |\mathbf{v}| |\nabla \varphi_1| |\varphi| dx ds. \tag{2.19}
\end{aligned}$$

We denote the four integrals on the right-hand side by $I_j(t)$, $1 \leq j \leq 4$, in that order, and estimate them individually. At first, it follows from (2.1) and Young's inequality that

$$I_1(t) \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \int_{\Omega} |\varphi|^2 dx ds. \tag{2.20}$$

Next, from the mean value theorem, (2.12), (2.8), and Young's inequality we infer that

$$\begin{aligned}
I_2(t) & \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \|\nabla w_1(s)\|_{L^\infty(Q)^3}^2 \|\varphi(s)\|_{L^2(\Omega)}^2 ds \\
& \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \int_{\Omega} |\varphi|^2 dx ds. \tag{2.21}
\end{aligned}$$

Moreover, (2.11) and Young's inequality imply that

$$\begin{aligned}
I_3(t) & \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \int_{\Omega} |\nabla w|^2 dx ds \\
& \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \int_{\Omega} |\varphi|^2 dx ds, \tag{2.22}
\end{aligned}$$

where the last inequality follows from **(H7)** and (2.16). Finally, we employ (2.7), Hölder's and Young's inequalities, as well as the continuity of the embedding $H^1(\Omega) \subset L^4(\Omega)$, to conclude that

$$\begin{aligned}
I_4(t) & \leq C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3} \|\nabla \varphi_1(s)\|_{L^4(\Omega)^3} \|\varphi(s)\|_{L^4(\Omega)} ds \\
& \leq \frac{c_0}{8} \int_0^t \|\varphi(s)\|_{H^1(\Omega)}^2 ds + C \int_0^t \|\varphi_1(s)\|_{H^2(\Omega)}^2 \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds \\
& \leq \frac{c_0}{8} \int_0^t \int_{\Omega} |\nabla \varphi|^2 dx ds + C \int_0^t \left(\|\varphi(s)\|_{L^2(\Omega)}^2 + \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 \right) ds. \tag{2.23}
\end{aligned}$$

Combining the estimates (2.19)–(2.23), and invoking Gronwall's lemma, we have thus shown that for any $t \in [0, T]$ we have

$$\max_{0 \leq s \leq t} \|\varphi(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\varphi(s)\|_{H^1(\Omega)}^2 ds \leq C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds, \tag{2.24}$$

where the constant C depends only on the data of the system and R .

Having established the stability estimate (2.24), we can now proceed to prove the stronger estimate (2.13). To this end, we multiply (2.14) by φ_t and integrate over $\Omega \times [0, t]$, where $t > 0$. Integration by parts and (2.18) yield that

$$\int_0^t \int_{\Omega} |\varphi_t|^2 dx ds + \frac{c_0}{2} \|\nabla \varphi(t)\|_{L^2(\Omega)^3}^2 \leq \sum_{i=1}^3 I_i(t), \quad (2.25)$$

where

$$\begin{aligned} I_1(t) &:= - \int_0^t \int_{\Omega} (m(\varphi_1) \nabla w_1 - m(\varphi_2) \nabla w_2) \cdot \nabla \varphi_t dx ds, \\ I_2(t) &:= - \int_0^t \int_{\Omega} (\mathbf{v} \cdot \nabla \varphi_1) \varphi_t dx ds, \\ I_3(t) &:= - \int_0^t \int_{\Omega} (\mathbf{v}_2 \cdot \nabla \varphi) \varphi_t dx ds. \end{aligned}$$

We estimate these expressions individually. The last two terms are easily handled. Indeed, owing to (2.7), Hölder's and Young's inequalities, and due to the continuity of the embedding $H^1(\Omega) \subset L^6(\Omega)$, we have, for any $\gamma > 0$ (to be specified later),

$$\begin{aligned} |I_2(t)| &\leq C \int_0^t \|\mathbf{v}(s)\|_{L^3(\Omega)^3} \|\nabla \varphi_1(s)\|_{L^6(\Omega)^3} \|\varphi_t(s)\|_{L^2(\Omega)} ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\varphi_t|^2 dx ds + \frac{C}{\gamma} \int_0^t \int_{\Omega} \|\varphi_1(s)\|_{H^2(\Omega)}^2 \|\mathbf{v}(s)\|_{L^3(\Omega)^3}^2 ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\varphi_t|^2 dx ds + \frac{C}{\gamma} \int_0^t \int_{\Omega} \|\mathbf{v}(s)\|_{L^3(\Omega)^3}^2 ds. \end{aligned} \quad (2.26)$$

Similarly, by also using (2.1) and (2.24), we obtain that

$$\begin{aligned} |I_3(t)| &\leq \int_0^t \int_{\Omega} \|\mathbf{v}_2(s)\|_{L^\infty(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)^3} \|\varphi_t(s)\|_{L^2(\Omega)} dx ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\varphi_t|^2 dx ds + \frac{C}{\gamma} \int_0^t \|\varphi(s)\|_{H^1(\Omega)}^2 ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\varphi_t|^2 dx ds + \frac{C}{\gamma} \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds. \end{aligned} \quad (2.27)$$

It remains to estimate the first integral. First notice that integration by parts with respect to time, together with (2.17), yields

$$\begin{aligned} I_1(t) &= - \int_{\Omega} ((m(\varphi_1) \nabla w_1 - m(\varphi_2) \nabla w_2) \cdot \nabla \varphi)(t) dx \\ &\quad + \int_0^t \int_{\Omega} (m(\varphi_1) \nabla w_1 - m(\varphi_2) \nabla w_2)_t \cdot \nabla \varphi dx ds =: I_{11}(t) + I_{12}(t). \end{aligned}$$

Using the mean value theorem, (2.8), (2.12), **(H7)**, and Young's inequality, we obtain

$$\begin{aligned}
|I_{11}(t)| &\leq \int_{\Omega} |m(\varphi_1(t)) - m(\varphi_2(t))| |\nabla w_1(t)| |\nabla \varphi(t)| \, dx \\
&\quad + \int_{\Omega} |m(\varphi_2(t))| |\nabla w(t)| |\nabla \varphi(t)| \, dx \\
&\leq C (\|\varphi(t)\|_{L^2(\Omega)} \|\nabla w_1(t)\|_{L^\infty(\Omega)^3} + \|\nabla w(t)\|_{L^2(\Omega)^3}) \|\nabla \varphi(t)\|_{L^2(\Omega)^3} \\
&\leq \gamma \|\nabla \varphi(t)\|_{L^2(\Omega)^3}^2 + \frac{C}{\gamma} \|\varphi(t)\|_{L^2(\Omega)}^2 \\
&\leq \gamma \|\nabla \varphi(t)\|_{L^2(\Omega)^3}^2 + \frac{C}{\gamma} \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 \, ds, \tag{2.28}
\end{aligned}$$

where the last inequality follows from (2.24).

Finally, we estimate $I_{12}(t)$. We have

$$\begin{aligned}
|I_{12}(t)| &\leq \int_0^t \int_{\Omega} |m'(\varphi_1) - m'(\varphi_2)| |\varphi_{1,t}| |\nabla w_1| |\nabla \varphi| \, dx \, ds \\
&\quad + \int_0^t \int_{\Omega} |m'(\varphi_2)| |\varphi_t| |\nabla w_1| |\nabla \varphi| \, dx \, ds \\
&\quad + \int_0^t \int_{\Omega} |m'(\varphi_2)| |\varphi_{2,t}| |\nabla w| |\nabla \varphi| \, dx \, ds \\
&\quad + \int_0^t \int_{\Omega} |m(\varphi_1) - m(\varphi_2)| |\nabla w_{1,t}| |\nabla \varphi| \, dx \, ds \\
&\quad + \int_0^t \int_{\Omega} |m(\varphi_2)| |\nabla w_t| |\nabla \varphi| \, dx \, ds \\
&=: J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \tag{2.29}
\end{aligned}$$

We estimate the terms on the right-hand side individually. First, invoking the mean value theorem, Hölder's inequality, the continuity of the embedding $H^1(\Omega) \subset L^4(\Omega)$, as well as (2.7), (2.8), (2.12) and (2.24), we find that

$$\begin{aligned}
|J_1(t)| &\leq C \int_0^t \|\varphi(s)\|_{L^4(\Omega)} \|\varphi_{1,t}(s)\|_{L^4(\Omega)} \|\nabla w_1(s)\|_{L^\infty(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)^3} \, ds \\
&\leq C \int_0^t \|\varphi(s)\|_{H^1(\Omega)}^2 \, ds \leq C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 \, ds. \tag{2.30}
\end{aligned}$$

Similarly, using (2.8), (2.12), (2.24) and Young's inequality, we have

$$\begin{aligned}
|J_2(t)| &\leq C \int_0^t \|m'(\varphi_2(s))\|_{L^\infty(\Omega)} \|\varphi_t(s)\|_{L^2(\Omega)} \|\nabla w_1(s)\|_{L^\infty(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)^3} \, ds \\
&\leq \gamma \int_0^t \int_{\Omega} |\varphi_t|^2 \, dx \, ds + \frac{C}{\gamma} \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 \, ds. \tag{2.31}
\end{aligned}$$

Also, invoking (2.7), (2.12), **(H7)**, the continuity of the embedding $H^1(\Omega) \subset L^4(\Omega)$ and (2.24), we find the estimate

$$\begin{aligned} |J_3(t)| &\leq \int_0^t \|m'(\varphi_2(s))\|_{L^\infty(\Omega)} \|\varphi_{2,t}(s)\|_{L^4(\Omega)} \|\nabla w(s)\|_{L^4(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)^3} ds \\ &\leq C \int_0^t \|\varphi(s)\|_{L^4(\Omega)} \|\varphi(s)\|_{H^1(\Omega)} ds \leq C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds. \end{aligned} \quad (2.32)$$

Similarly, again using (2.7), (2.12), (2.24), together with Hölder's inequality and **(H7)**, we obtain that

$$\begin{aligned} |J_4(t)| &\leq C \int_0^t \|\varphi(s)\|_{L^4(\Omega)} \|\nabla w_{1,t}(s)\|_{L^4(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)} ds \\ &\leq C \int_0^t \|\varphi(s)\|_{H^1(\Omega)}^2 \|\varphi_{1,t}(s)\|_{L^4(\Omega)^3} ds \leq C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds. \end{aligned} \quad (2.33)$$

Finally, we conclude from (2.12), **(H7)**, (2.24), as well as Hölder's and Young's inequalities, that

$$\begin{aligned} |J_5(t)| &\leq C \int_0^t \|\nabla w_t(s)\|_{L^2(\Omega)^3} \|\nabla \varphi(s)\|_{L^2(\Omega)^3} ds \\ &\leq C \int_0^t \|\varphi_t(s)\|_{L^2(\Omega)} \|\varphi(s)\|_{H^1(\Omega)} ds \\ &\leq \gamma \int_0^t \int_\Omega |\varphi_t|^2 dx ds + \frac{C}{\gamma} \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}^2 ds. \end{aligned} \quad (2.34)$$

Combining the estimates (2.25)–(2.34), observing the continuity of the embedding $L^3(\Omega) \subset L^2(\Omega)$, and adjusting $\gamma > 0$ appropriately small, we have finally shown the assertion. \blacksquare

Remark 3. By virtue of **(H7)** and **(H8)**, the stability estimates (2.13) and (2.24) entail corresponding estimates for w and μ . In particular, we may without loss of generality assume (by possibly choosing an appropriately larger $K_2^* > 0$) that for all $t \in [0, T]$ we have

$$\int_0^t \|\nabla w_t(s)\|_{L^2(\Omega)^3}^2 ds + \|w\|_{L^\infty(0,T;H^2(\Omega))}^2 \leq K_2^* \int_0^t \|\mathbf{v}(s)\|_{L^3(\Omega)^3}^2 ds, \quad (2.35)$$

$$\int_0^t \|\nabla \mu_t(s)\|_{L^2(\Omega)^3}^2 ds + \|\mu\|_{L^\infty(0,T;H^2(\Omega))}^2 \leq K_2^* \int_0^t \|\mathbf{v}(s)\|_{L^3(\Omega)^3}^2 ds. \quad (2.36)$$

3 Optimal control

In this section, we study the optimal control problem **(CP)** with \mathcal{V}_{ad} defined as in **(H1)**, and we assume that the general assumptions **(H2)**–**(H8)** are satisfied. Notice that, owing to Lemma 2.1, the *control-to-state operator*

$$\mathcal{S} : \mathcal{V}_R \rightarrow W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega)); \quad \mathbf{v} \mapsto \varphi,$$

is well defined and Lipschitz continuous as a mapping from \mathcal{V}_R (viewed as a subset of $L^2(0, T; L^3(\Omega)^3)$) into $H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega))$. Moreover, all of the global bounds (2.7), (2.8), (2.9), (2.10), (2.12), as well as all of the stability estimates (2.13), (2.25), (2.35) and (2.36), are satisfied.

We are now ready to prove existence for the control problem **(CP)**.

Theorem 3.1. *Suppose that the hypotheses **(H1)**–**(H8)** are fulfilled. Then the problem **(CP)** admits a solution $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$.*

PROOF. Let $\{\mathbf{v}_n\} \subset \mathcal{V}_{ad}$ be a minimizing sequence for **(CP)** and $\varphi_n = \mathcal{S}(\mathbf{v}_n)$, $n \in \mathbb{N}$. Then it follows from **(H1)** and (2.7) that there exist $(\bar{\mathbf{v}}, \bar{\varphi})$ such that, possibly for a subsequence which is again indexed by n , we have

$$\begin{aligned} \mathbf{v}_n &\rightarrow \bar{\mathbf{v}} \quad \text{weakly star in } L^2(0, T; H_{div}^1(\Omega) \cap L^\infty(\Omega)^3) \cap H^1(0, T; L^3(\Omega)^3), \\ \varphi_n &\rightarrow \bar{\varphi} \quad \text{weakly star in } W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)). \end{aligned}$$

Clearly, $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$. In addition, by virtue of standard compactness lemmas (cf. [23, Thm. 5.1, p. 58] and [30, Sec. 8, Cor. 4]), we have the strong convergences

$$\begin{aligned} \mathbf{v}_n &\rightarrow \bar{\mathbf{v}} \quad \text{strongly in } C^0([0, T]; L^2(\Omega)^3), \\ \varphi_n &\rightarrow \bar{\varphi} \quad \text{strongly in } C^0([0, T]; H^s(\Omega)) \quad \forall s \in [0, 2), \end{aligned}$$

which implies, in particular, that

$$\varphi_n \rightarrow \bar{\varphi} \quad \text{strongly in } C^0(\bar{Q}).$$

Owing to the separation property (2.6) and the assumptions on f and m , we also have

$$f'(\varphi_n) \rightarrow f'(\bar{\varphi}) \quad \text{and} \quad m(\varphi_n) \rightarrow m(\bar{\varphi}), \quad \text{both strongly in } C^0(\bar{Q}).$$

Finally, it is easily deduced from **(H6)** that $\{w_n := \mathcal{K}(\varphi_n)\}$ converges strongly in $C^0(\bar{Q})$ to $\bar{w} := \mathcal{K}(\bar{\varphi})$ (recall (1.4) and (1.15)). In summary, we can pass to the limit as $n \rightarrow \infty$ in (1.2)–(1.6), written for $(\mathbf{v}_n, \varphi_n)$, finding that $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$; i.e., the pair $(\bar{\mathbf{v}}, \bar{\varphi})$ is admissible for **(CP)**. It then follows from the weak sequential lower semicontinuity properties of J that \bar{u} , together with the associated state $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$, is a solution to **(CP)**. ■

We now turn our interest to the derivation of necessary first-order optimality conditions for problem **(CP)**. Referring to [31] for a detailed discussion and description of the various techniques related to optimality conditions, we proceed as follows: we first prove a suitable differentiability property for the control-to-state operator \mathcal{S} , using the *linearized* system, and then we establish the necessary optimality conditions in terms of a *variational inequality* and the associated *adjoint state equation*. In the following, we will always (unless it is explicitly stated otherwise) assume that $\bar{\mathbf{v}} \in \mathcal{V}_R$ is fixed and that $(\bar{\varphi}, \bar{w}, \bar{\mu})$ is the associated triple solving the state system, i.e.,

$$\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}}), \quad \bar{w} = \mathcal{K}(\bar{\varphi}), \quad \bar{\mu} = f'(\bar{\varphi}) + \bar{w}.$$

The linearized system. Suppose that an arbitrary $\mathbf{h} \in \mathcal{V}$ is given. As a preparation for the proof of differentiability, we consider the following system, which is obtained by linearizing the state system (1.2)–(1.6) at $\bar{\varphi} = S(\bar{\mathbf{v}})$:

$$\begin{aligned} \xi_t - c_0 \Delta \xi - \operatorname{div} \left(m'(\bar{\varphi}) \xi \nabla \bar{w} - 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|) \xi(y, \cdot) dy \right) \right) \\ = -\mathbf{h} \cdot \nabla \bar{\varphi} - \bar{\mathbf{v}} \cdot \nabla \xi \quad \text{a.e. in } Q, \end{aligned} \quad (3.1)$$

$$\bar{w}(x, t) = \int_{\Omega} k(|x-y|)(1-2\bar{\varphi}(y, t)) dy \quad \text{a.e. in } Q, \quad (3.2)$$

$$\begin{aligned} \left(c_0 \nabla \xi + m'(\bar{\varphi}) \xi \nabla \bar{w} - 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|) \xi(y, \cdot) dy \right) \right) \cdot \mathbf{n} = 0 \\ \text{a.e. on } \Sigma, \end{aligned} \quad (3.3)$$

$$\xi(0) = 0 \quad \text{a.e. in } \Omega. \quad (3.4)$$

After proving that (3.1)–(3.4) has a unique solution ξ , we expect that $\xi = DS(\bar{\mathbf{v}})\mathbf{h}$, where $DS(\bar{\mathbf{v}})$ denotes the Fréchet derivative of S at $\bar{\mathbf{v}}$. Recalling the global bounds (2.7)–(2.12), we can expect the regularity

$$\xi \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)). \quad (3.5)$$

We have the following result.

Proposition 3.2. *Suppose that the hypotheses (H1)–(H8) are fulfilled. Then the problem (3.1)–(3.4) has a unique solution satisfying (3.5).*

PROOF. The proof is performed via a Faedo–Galerkin scheme inspired by [6, Sec. 4]. To this end, we choose $\{\psi_j\}_{j \in \mathbb{N}}$ to be the family of (appropriately orthonormalized and ordered) eigenfunctions to the eigenvalue problem

$$-\Delta \psi + \psi = \lambda \psi \quad \text{in } \Omega, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0 \quad \text{on } \partial \Omega,$$

as a Galerkin base in $H^1(\Omega)$. Putting $\xi_n(x, t) := \sum_{k=1}^n a_k(t) \psi_k(x)$, we then look for a solution to the approximating problem

$$\begin{aligned} \int_{\Omega} \xi_n'(t) \psi dx + \int_{\Omega} c_0 \nabla \xi_n(t) \cdot \nabla \psi dx + \int_{\Omega} \left(m'(\bar{\varphi}(t)) \xi_n(t) \nabla \bar{w}(t) \right. \\ \left. - 2m(\bar{\varphi}(t)) \nabla \left(\int_{\Omega} k(|x-y|) \xi_n(y, t) dy \right) \right) \cdot \nabla \psi dx \\ = - \int_{\Omega} (\mathbf{h}(t) \cdot \nabla \bar{\varphi}(t)) \psi dx - \int_{\Omega} (\bar{\mathbf{v}}(t) \cdot \nabla \xi_n(t)) \psi dx, \quad \text{for } t \in (0, T], \end{aligned} \quad (3.6)$$

$$\xi_n(0) = 0 \quad \text{a.e. in } \Omega, \quad (3.7)$$

for every $\psi \in \Psi_n := \text{span} \{\psi_1, \dots, \psi_n\}$. Apparently, this is nothing but an initial value problem for a system of linear ordinary differential equations for the unknown functions a_1, \dots, a_n , where, owing to the global bounds (2.7)–(2.12), all occurring coefficient functions are continuous on $[0, T]$. It is therefore a standard matter to show that there exists some $T_n \in (0, T]$ such that the ODE system has a maximal solution $\mathbf{a} := (a_1, \dots, a_n) \in C^1([0, T_n]; \mathbb{R}^n)$ that specifies a solution $\xi_n \in C^1([0, T_n]; H^3(\Omega))$. Observe that

$$\frac{\partial \xi_n}{\partial \mathbf{n}} = \frac{\partial \Delta \xi_n}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma \quad \forall n \in \mathbb{N}. \quad (3.8)$$

We now aim to prove a (uniform in $n \in \mathbb{N}$) estimate for ξ_n in $C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$. Once this is shown, it is a standard matter to show that $T_n = T$ and to pass to the limit as $n \rightarrow \infty$ to recover a solution with the asserted regularity to the linearized problem (3.1)–(3.4). Due to regularity of the coefficients and of the known functions in the system, we can also prove that the solution is unique, simply by testing the difference between two equations (3.1), written for two possible different solutions ξ_1 and ξ_2 , by $\xi_1 - \xi_2$ and then exploiting the linearity of the problem. Since these arguments are straightforward, we can afford to be brief here and to restrict ourselves to the derivation of the asserted global bounds.

To this end, let $t \in (0, T_n)$ be arbitrary. In what follows, C_i , $i \in \mathbb{N}$, will denote positive constants that may depend on the data of the system but not on $n \in \mathbb{N}$, and we will make repeated use of the global bounds (2.7)–(2.12) and of (2.1). First observe that the global bounds and hypothesis **(H7)** imply that, for any $s \in [0, t]$,

$$\begin{aligned} & \left\| m'(\bar{\varphi}(s)) \xi_n(s) \nabla \bar{w}(s) - 2m(\bar{\varphi}(s)) \nabla \left(\int_{\Omega} k(|x-y|) \xi_n(y, s) dy \right) \right\|_{L^2(\Omega)} \\ & \leq C_1 \|\xi_n(s)\|_{L^2(\Omega)}. \end{aligned} \quad (3.9)$$

Now we insert $\psi = \xi_n(t)$ in (3.6) and integrate over $[0, t]$ to find that

$$\frac{1}{2} \|\xi_n(t)\|_{L^2(\Omega)}^2 + c_0 \int_0^t \int_{\Omega} |\nabla \xi_n|^2 dx ds \leq \sum_{j=1}^3 I_j(t), \quad (3.10)$$

with expressions $I_j(t)$, $1 \leq j \leq 3$, that will be specified and estimated below.

Let $\gamma > 0$ be arbitrary (to be specified later). We have, using (3.9) and Young's inequality,

$$\begin{aligned} |I_1(t)| &= \left| \int_0^t \int_{\Omega} \left[m'(\bar{\varphi}) \xi_n \nabla \bar{w} - 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|) \xi_n(y, s) dy \right) \right] \cdot \nabla \xi_n dx ds \right| \\ &\leq \gamma \int_0^t \int_{\Omega} |\nabla \xi_n|^2 dx ds + \frac{C_2}{\gamma} \int_0^t \int_{\Omega} |\xi_n|^2 dx ds. \end{aligned} \quad (3.11)$$

Moreover,

$$\begin{aligned} |I_2(t)| &= \left| \int_0^t \int_{\Omega} (\mathbf{h} \cdot \nabla \bar{\varphi}) \xi_n dx ds \right| \leq \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3} \|\nabla \bar{\varphi}(s)\|_{L^6(\Omega)^3} \|\xi_n(s)\|_{L^2(\Omega)} ds \\ &\leq C_3 \left(\int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3}^2 ds + \int_0^t \int_{\Omega} |\xi_n|^2 dx ds \right), \end{aligned} \quad (3.12)$$

as well as

$$\begin{aligned}
|I_3(t)| &= \left| \int_0^t \int_{\Omega} (\bar{\mathbf{v}} \cdot \nabla \xi_n) \xi_n \, dx \, ds \right| \\
&\leq C_4 \int_0^t \|\bar{\mathbf{v}}(s)\|_{L^\infty(\Omega)^3} \|\nabla \xi_n(s)\|_{L^2(\Omega)^3} \|\xi_n(s)\|_{L^2(\Omega)} \, ds \\
&\leq \gamma \int_0^t \int_{\Omega} |\nabla \xi_n|^2 \, dx \, ds + \frac{C_5}{\gamma} \int_0^t \int_{\Omega} |\xi_n|^2 \, dx \, ds. \tag{3.13}
\end{aligned}$$

Combining the estimates (3.10)–(3.13), choosing $\gamma > 0$ small enough, and applying Gronwall's lemma, we have thus shown the estimate

$$\max_{0 \leq s \leq t} \|\xi_n(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds \leq C_6 \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3}^2 \, ds \leq C_7. \tag{3.14}$$

Next, we insert $\psi = -\Delta \xi_n(t)$ in (3.6), integrate by parts using the boundary condition (3.8), and then integrate over $[0, t]$. We then obtain

$$\begin{aligned}
\frac{1}{2} \|\nabla \xi_n(t)\|_{L^2(\Omega)^3}^2 + c_0 \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds &\leq \int_0^t \int_{\Omega} (|g_1| + |g_2|) |\Delta \xi_n| \, dx \, ds \\
+ \int_0^t \int_{\Omega} |\mathbf{h}| |\nabla \bar{\varphi}| |\Delta \xi_n| \, dx \, ds + \int_0^t \int_{\Omega} |\bar{\mathbf{v}}| |\nabla \xi_n| |\Delta \xi_n| \, dx \, ds, \tag{3.15}
\end{aligned}$$

where the functions g_1, g_2 will be specified and estimated below. Now let $\gamma > 0$ be arbitrary (to be specified later). The last two integrals on the right-hand side of (3.15) are easily estimated. In fact, using the general bounds (2.7), as well as Hölder's and Young's inequalities and (2.1), we have

$$\begin{aligned}
\int_0^t \int_{\Omega} |\mathbf{h}| |\nabla \bar{\varphi}| |\Delta \xi_n| \, dx \, ds &\leq C_8 \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3} \|\nabla \bar{\varphi}(s)\|_{L^6(\Omega)^3} \|\Delta \xi_n(s)\|_{L^2(\Omega)} \, ds \\
&\leq \gamma \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds + \frac{C_9}{\gamma} \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3}^2 \, ds, \tag{3.16}
\end{aligned}$$

as well as

$$\begin{aligned}
\int_0^t \int_{\Omega} |\bar{\mathbf{v}}| |\nabla \xi_n| |\Delta \xi_n| \, dx \, ds &\leq \int_0^t \|\bar{\mathbf{v}}(s)\|_{L^\infty(\Omega)^3} \|\nabla \xi_n(s)\|_{L^2(\Omega)^3} \|\Delta \xi_n(s)\|_{L^2(\Omega)} \, ds \\
&\leq \gamma \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds + \frac{C_{10}}{\gamma} \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds. \tag{3.17}
\end{aligned}$$

It remains to estimate the first integral on the right-hand side of (3.15). To this end, we first infer from the global bounds (2.7)–(2.12) that a.e. on Q it holds

$$|g_1| := |\operatorname{div} [m'(\bar{\varphi}) \xi_n \nabla \bar{w}]| \leq C_{11} \left(|\xi_n| (|\nabla \bar{\varphi}| + |\Delta \bar{w}|) + |\nabla \xi_n| |\nabla \bar{\varphi}| \right),$$

where it is easily verified that the expression in the inner bracket, which we denote by z , is bounded in $C^0([0, T]; L^2(\Omega))$. We thus have, invoking (1.10) and (1.14),

$$\begin{aligned} \int_0^t \int_{\Omega} |\xi_n| |z| |\Delta \xi_n| \, dx \, ds &\leq \int_0^t \|\xi_n(s)\|_{L^\infty(\Omega)} \|z(s)\|_{L^2(\Omega)} \|\Delta \xi_n(s)\|_{L^2(\Omega)} \, ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds + \frac{C_{12}}{\gamma} \int_0^t \|\xi_n(s)\|_{L^\infty(\Omega)}^2 \, ds \\ &\leq 2\gamma \int_0^t \|\xi_n(s)\|_{H^2(\Omega)}^2 \, ds + C_{13} (\gamma^{-1} + \gamma^{-3}) \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds. \end{aligned} \quad (3.18)$$

Moreover, by (2.7), (1.13), and Hölder's and Young's inequalities, it holds

$$\begin{aligned} \int_0^t \int_{\Omega} |\nabla \bar{\varphi}| |\nabla \xi_n| |\Delta \xi_n| \, dx \, ds &\leq C_{14} \int_0^t \|\nabla \bar{\varphi}(s)\|_{L^6(\Omega)^3} \|\nabla \xi_n(s)\|_{L^3(\Omega)^3} \|\Delta \xi_n(s)\|_{L^2(\Omega)} \, ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds + \frac{C_{15}}{\gamma} \int_0^t \|\nabla \xi_n(s)\|_{L^3(\Omega)^3}^2 \, ds \\ &\leq 2\gamma \int_0^t \|\xi_n(s)\|_{H^2(\Omega)}^2 \, ds \\ &\quad + C_{16} (\gamma^{-1} + \gamma^{-3}) \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds. \end{aligned} \quad (3.19)$$

Finally, notice that for a.e. $(x, t) \in Q$ we have

$$|g_2(x, t)| := \left| \operatorname{div} \left[2m(\bar{\varphi}(x, t)) \nabla \left(\int_{\Omega} k(|x-y|) \xi_n(y, t) \, dy \right) \right] \right|,$$

and it easily follows from (2.7), (2.12), and the hypotheses **(H7)** and **(H8)**, that

$$\int_0^t \|g_2(s)\|_{L^2(\Omega)}^2 \, ds \leq C_{17} \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds,$$

whence we obtain that

$$\int_0^t \int_{\Omega} |g_2| |\Delta \xi_n| \, dx \, ds \leq \gamma \int_0^t \int_{\Omega} |\Delta \xi_n|^2 \, dx \, ds + \frac{C_{18}}{\gamma} \int_0^t \|\xi_n(s)\|_{H^1(\Omega)}^2 \, ds. \quad (3.20)$$

Now observe that $\partial \xi_n / \partial \mathbf{n} = 0$, so that standard elliptic estimates imply that

$$\|\xi_n(s)\|_{H^2(\Omega)} \leq C_{19} (\|\Delta \xi_n(s)\|_{L^2(\Omega)} + \|\xi_n(s)\|_{L^2(\Omega)}),$$

where $C_{19} > 0$ depends only on Ω . Therefore, choosing $\gamma > 0$ appropriately small, and invoking (3.14), we can infer from the estimates (3.15)–(3.20) that

$$\max_{0 \leq s \leq t} \|\xi_n(s)\|_{H^1(\Omega)}^2 + \int_0^t \|\xi_n(s)\|_{H^2(\Omega)}^2 \, ds \leq C_{20} \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3}^2 \, ds \leq C_{21}. \quad (3.21)$$

This concludes the proof of the assertion. \blacksquare

Remark 4. From (3.21) it follows, in particular, that the linear mapping $\mathbf{h} \mapsto \xi =: \xi^{\mathbf{h}}$ is continuous as a mapping from \mathcal{V} into the space $C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$.

Differentiability of the control-to-state mapping. In this section we are going to prove the following result:

Proposition 3.3. *Let the hypotheses (H1)–(H8) be satisfied. Then the control-to-state operator*

$$\mathcal{S} : \mathcal{V}_R \rightarrow W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap C^0([0, T]; H^2(\Omega)), \quad \mathbf{v} \mapsto \varphi,$$

is Fréchet differentiable in \mathcal{V}_R as a mapping from \mathcal{V} into $\mathcal{Y} := C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, and for every $\bar{\mathbf{v}} \in \mathcal{V}_R$ the Fréchet derivative $D\mathcal{S}(\bar{\mathbf{v}}) \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$ is given by the relation

$$D\mathcal{S}(\bar{\mathbf{v}})\mathbf{h} = \xi^{\mathbf{h}}, \quad (3.22)$$

where $\xi^{\mathbf{h}}$ is the unique solution to the linearized system (3.1)–(3.4) with $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$.

PROOF. Let $\bar{\mathbf{v}} \in \mathcal{V}_R$ be fixed, and let $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$. Since \mathcal{V}_R is open, there is some $\Lambda > 0$ such that $\bar{\mathbf{v}} + \mathbf{h} \in \mathcal{V}_R$ whenever $\|\mathbf{h}\|_{\mathcal{V}} \leq \Lambda$. In the following, we only consider such perturbations \mathbf{h} and set

$$\mathbf{v}^{\mathbf{h}} = \bar{\mathbf{v}} + \mathbf{h}, \quad \varphi^{\mathbf{h}} = \mathcal{S}(\mathbf{v}^{\mathbf{h}}), \quad y^{\mathbf{h}} = \varphi^{\mathbf{h}} - \bar{\varphi} - \xi^{\mathbf{h}}.$$

Since the linear mapping $\mathbf{h} \mapsto \xi^{\mathbf{h}}$ is by Remark 4 continuous as a mapping from \mathcal{V} into \mathcal{Y} , it suffices to show that there exists an increasing mapping $Z : (0, +\infty) \rightarrow (0, +\infty)$ such that $\lim_{\lambda \searrow 0} Z(\lambda)/\lambda^2 = 0$ and

$$\|y^{\mathbf{h}}\|_{C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))}^2 \leq Z(\|\mathbf{h}\|_{\mathcal{V}}). \quad (3.23)$$

In the following, we will denote by C_i , $i \in \mathbb{N}$, positive constants that may depend on the data of the system and on R , but not on the special choice of $\mathbf{h} \in \mathcal{V}$ with $\|\mathbf{h}\|_{\mathcal{V}} \leq \Lambda$. For a shorter exposition, we also often omit the arguments of the involved functions if no confusion may arise. Notice that the global bounds (2.7), (2.9) and (2.12) are satisfied by $\varphi^{\mathbf{h}}$ for any perturbation \mathbf{h} with $\|\mathbf{h}\|_{\mathcal{V}} \leq \Lambda$, and, owing to the weak sequential lower semicontinuity of norms, it follows from (3.21) that for all such perturbations we have

$$\|\xi^{\mathbf{h}}\|_{C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))} \leq C_1. \quad (3.24)$$

First of all, it is easily verified that $y^{\mathbf{h}}$ is a “strong” solution to the following system:

$$\begin{aligned} y_t^{\mathbf{h}} - c_0 \Delta y^{\mathbf{h}} - \operatorname{div} \left[m(\varphi^{\mathbf{h}}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\varphi^{\mathbf{h}}) dy \right) \right. \\ \quad - m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\ \quad - m'(\bar{\varphi}) \xi^{\mathbf{h}} \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\ \quad \left. + 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|) \xi^{\mathbf{h}}(y) dy \right) \right] \\ + \mathbf{v} \cdot \nabla y^{\mathbf{h}} + \mathbf{h} \cdot (\nabla \varphi^{\mathbf{h}} - \nabla \bar{\varphi}) = 0 \quad \text{a.e. in } Q, \end{aligned} \quad (3.25)$$

$$\begin{aligned}
& \left[c_0 \nabla y^{\mathbf{h}} + m(\varphi^{\mathbf{h}}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\varphi^{\mathbf{h}}) dy \right) \right. \\
& \quad - m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\
& \quad - m'(\bar{\varphi}) \xi^{\mathbf{h}} \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\
& \quad \left. + 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)\xi^{\mathbf{h}}(y) dy \right) \right] \cdot \mathbf{n} = 0 \quad \text{a.e. on } \Sigma, \quad (3.26)
\end{aligned}$$

$$y^{\mathbf{h}}(0) = 0 \quad \text{a.e. in } \Omega. \quad (3.27)$$

We now test (3.25) by $y^{\mathbf{h}}$, integrate over $(0, t)$ where $t \in (0, T]$, and use (3.26) and (3.27) to get

$$\begin{aligned}
& \frac{1}{2} \|y^{\mathbf{h}}(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} |\nabla y^{\mathbf{h}}|^2 dx ds \\
& + \int_0^t \int_{\Omega} \nabla y^{\mathbf{h}} \cdot \left\{ m(\varphi^{\mathbf{h}}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\varphi^{\mathbf{h}}) dy \right) \right. \\
& \quad - m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\
& \quad - m'(\bar{\varphi}) \xi^{\mathbf{h}} \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}) dy \right) \\
& \quad \left. + 2m(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)\xi^{\mathbf{h}}(y) dy \right) \right\} dx ds \\
& + \int_0^t \int_{\Omega} y^{\mathbf{h}}(\bar{\mathbf{v}} \cdot \nabla y^{\mathbf{h}}) dx ds + \int_0^t \int_{\Omega} \mathbf{h} \cdot (\nabla \varphi^{\mathbf{h}} - \nabla \bar{\varphi}) y^{\mathbf{h}} dx ds = 0. \quad (3.28)
\end{aligned}$$

We have $\int_0^t \int_{\Omega} y^{\mathbf{h}}(\bar{\mathbf{v}} \cdot \nabla y^{\mathbf{h}}) dx ds = 0$, since $\bar{\mathbf{v}}$ vanishes on $\partial\Omega$ and is divergence free. Moreover, using Hölder's and Young's inequalities, as well as the stability estimate (2.13) and the continuity of the embedding $H^1(\Omega) \subset L^6(\Omega)$, we obtain that

$$\begin{aligned}
& \left| \int_0^t \int_{\Omega} \mathbf{h} \cdot (\nabla \varphi^{\mathbf{h}} - \nabla \bar{\varphi}) y^{\mathbf{h}} dx ds \right| \\
& \leq \int_0^t \|\nabla(\varphi^{\mathbf{h}} - \bar{\varphi})(s)\|_{L^2(\Omega)^3} \|y^{\mathbf{h}}(s)\|_{L^6(\Omega)} \|\mathbf{h}(s)\|_{L^3(\Omega)^3} ds \\
& \leq \gamma \int_0^t \|y^{\mathbf{h}}(s)\|_{H^1(\Omega)}^2 ds + \frac{C_2}{\gamma} \int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)}^2 \|\nabla(\varphi^{\mathbf{h}} - \bar{\varphi})(s)\|_{L^2(\Omega)}^2 ds \\
& \leq \gamma \int_0^t \|y^{\mathbf{h}}(s)\|_{H^1(\Omega)}^2 ds + \frac{C_3}{\gamma} \left(\int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)}^2 ds \right)^2, \quad (3.29)
\end{aligned}$$

for every positive γ (to be chosen later).

It remains to estimate the third summand in (3.28). To this end, we observe that the expression

in the curly bracket in (3.28) equals the sum of the following three expressions:

$$\begin{aligned} A_1(x, s) &:= (m(\varphi^{\mathbf{h}}) - m(\bar{\varphi}) - m'(\bar{\varphi})\xi^{\mathbf{h}})(x, s) \nabla \int_{\Omega} k(|x - y|) (1 - 2\bar{\varphi}(y, s)) dy, \\ A_2(x, s) &:= -2(m(\varphi^{\mathbf{h}}) - m(\bar{\varphi}))(x, s) \nabla \int_{\Omega} k(|x - y|)(\varphi^{\mathbf{h}}(y, s) - \bar{\varphi}(y, s)) dy, \\ A_3(x, s) &:= -2m(\bar{\varphi}(x, s)) \nabla \int_{\Omega} k(|x - y|) y^{\mathbf{h}}(y, s) dy. \end{aligned}$$

Moreover, Taylor's theorem, using also the separation property (2.6) and the global bounds (2.12), yields that almost everywhere in Q it holds

$$\begin{aligned} |m(\varphi^{\mathbf{h}}) - m(\bar{\varphi}) - m'(\bar{\varphi})\xi^{\mathbf{h}}| &\leq C_4 |y^{\mathbf{h}}| + \frac{1}{2} \max_{\kappa \leq \sigma \leq 1-\kappa} |m''(\sigma)| |\varphi^{\mathbf{h}} - \bar{\varphi}|^2 \\ &\leq C_4 |y^{\mathbf{h}}| + C_5 |\varphi^{\mathbf{h}} - \bar{\varphi}|^2. \end{aligned} \quad (3.30)$$

Now, by virtue of hypothesis **(H7)** and (2.8), and by invoking Hölder's and Young's inequalities, we have

$$\begin{aligned} \int_0^t \int_{\Omega} |A_1| |\nabla y^{\mathbf{h}}| dx ds &\leq C_6 \int_0^t \int_{\Omega} (|y^{\mathbf{h}}| + |\varphi^{\mathbf{h}} - \bar{\varphi}|^2) |\nabla y^{\mathbf{h}}| dx ds \\ &\leq C_7 \int_0^t \int_{\Omega} (|y^{\mathbf{h}}| + |\varphi^{\mathbf{h}} - \bar{\varphi}|^2) |\nabla y^{\mathbf{h}}| dx ds \\ &\leq \int_0^t \int_{\Omega} (\gamma |\nabla y^{\mathbf{h}}|^2 + \frac{C_8}{\gamma} |y^{\mathbf{h}}|^2) dx ds + \int_0^t (\|\varphi^{\mathbf{h}} - \bar{\varphi}\|_{L^4(\Omega)}^2 \|\nabla y^{\mathbf{h}}\|_{L^2(\Omega^3)}^2)(s) ds \\ &\leq 2\gamma \int_0^t \int_{\Omega} |\nabla y^{\mathbf{h}}|^2 dx ds + \frac{C_9}{\gamma} \left(\int_0^t \int_{\Omega} |y^{\mathbf{h}}|^2 dx ds + \left(\int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega^3)}^2 ds \right)^2 \right), \end{aligned} \quad (3.31)$$

where again (2.13) was employed. Similarly,

$$\begin{aligned} \int_0^t \int_{\Omega} |A_2| |\nabla y^{\mathbf{h}}| dx ds &\leq C_{10} \int_0^t \|\nabla y^{\mathbf{h}}(s)\|_{L^2(\Omega^3)} \|\varphi^{\mathbf{h}}(s) - \bar{\varphi}(s)\|_{L^4(\Omega)}^2 ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\nabla y^{\mathbf{h}}|^2 dx ds + \frac{C_{11}}{\gamma} \int_0^t \|\varphi^{\mathbf{h}}(s) - \bar{\varphi}(s)\|_{L^4(\Omega)}^4 ds, \end{aligned} \quad (3.32)$$

as well as, using hypothesis **(H7)** once more,

$$\begin{aligned} \int_0^t \int_{\Omega} |A_3| |\nabla y^{\mathbf{h}}| dx ds &\leq C_{12} \int_0^t \|\nabla y^{\mathbf{h}}(s)\|_{L^2(\Omega^3)} \left\| \nabla \int_{\Omega} k(|x - y|) y^{\mathbf{h}}(y, s) dy \right\|_{L^2(\Omega^3)} ds \\ &\leq \gamma \int_0^t \int_{\Omega} |\nabla y^{\mathbf{h}}|^2 dx ds + \frac{C_{13}}{\gamma} \int_0^t \int_{\Omega} |y^{\mathbf{h}}|^2 dx ds. \end{aligned} \quad (3.33)$$

Collecting the estimates (3.28), (3.29) and (3.31)–(3.33), choosing $\gamma > 0$ small enough, and invoking the stability estimate (2.13), we can finally conclude from Gronwall's lemma that

$$\|y^{\mathbf{h}}\|_{C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))}^2 \leq C_{14} \left(\int_0^T \|\mathbf{h}(t)\|_{L^3(\Omega^3)}^2 dt \right)^2 \leq C_{15} \|\mathbf{h}\|_{\mathcal{V}}^4. \quad (3.34)$$

This concludes the proof of Proposition 3.3. ■

Using the convexity of \mathcal{V}_{ad} , we immediately conclude from Proposition 3.3 the following result.

Corollary 3.4. *Assume that the hypotheses **(H1)–(H8)** are fulfilled, and let $\bar{\mathbf{v}} \in \mathcal{V}_{\text{ad}}$ be an optimal control for problem **(CP)** with associated state $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$. Then we have for every $\mathbf{v} \in \mathcal{V}_{\text{ad}}$ the inequality*

$$\begin{aligned} & \beta_1 \int_0^T \int_{\Omega} (\bar{\varphi} - \varphi_Q) \xi^{\mathbf{h}} \, dx \, ds + \beta_2 \int_{\Omega} (\bar{\varphi}(T) - \varphi_{\Omega}) \xi^{\mathbf{h}}(T) \, dx \\ & + \beta_3 \int_0^T \int_{\Omega} \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) \, dx \, ds \geq 0, \end{aligned} \quad (3.35)$$

where $\xi^{\mathbf{h}}$ is the unique solution to the linearized system (3.1)–(3.4) associated with $\mathbf{h} = \mathbf{v} - \bar{\mathbf{v}}$.

The adjoint system and first order necessary optimality conditions. In order to establish the necessary first order optimality conditions for **(CP)**, we need to eliminate $\xi^{\mathbf{h}}$ from inequality (3.35). To this end, we introduce the *adjoint system* which formally reads as follows:

$$\begin{aligned} & -p_t - c_0 \Delta p - \nabla p \cdot \left[\bar{\mathbf{v}} + m'(\bar{\varphi}) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}(y,t)) \, dy \right) \right] \\ & - 2 \int_{\Omega} \nabla p(y,t) m(\bar{\varphi}(y,t)) \cdot \nabla k(|x-y|) \, dy = \beta_1 (\bar{\varphi} - \varphi_Q) \quad \text{in } Q, \end{aligned} \quad (3.36)$$

$$\frac{\partial p}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma, \quad (3.37)$$

$$p(T) = \beta_2 (\bar{\varphi}(T) - \varphi_{\Omega}) \quad \text{a.e. in } \Omega. \quad (3.38)$$

Since the final value $p(T)$ only belongs to $L^2(\Omega)$, we can at best expect the regularity

$$p \in H^1(0, T; H^1(\Omega)^*) \cap C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),$$

which entails that (3.36)–(3.37) must be understood in the weak variational sense. To this end, we denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$. We then rewrite (3.36)–(3.37) in the form

$$\begin{aligned} & \langle p_t(t), \eta \rangle + c_0 \int_{\Omega} \nabla p(t) \cdot \nabla \eta \, dx \\ & - \int_{\Omega} \eta \nabla p(t) \cdot \left[\bar{\mathbf{v}}(t) + m'(\bar{\varphi}(t)) \nabla \left(\int_{\Omega} k(|x-y|)(1-2\bar{\varphi}(y,t)) \, dy \right) \right] \, dx \\ & - 2 \int_{\Omega} \eta \int_{\Omega} \nabla p(y,t) m(\bar{\varphi}(y,t)) \cdot \nabla k(|x-y|) \, dy \, dx = \int_{\Omega} \eta \beta_1 (\bar{\varphi}(t) - \varphi_Q(t)) \, dx, \end{aligned} \quad (3.39)$$

for every $\eta \in H^1(\Omega)$ and almost every $t \in (0, T)$.

We have the following existence and uniqueness result.

Proposition 3.5. *The adjoint system (3.36)–(3.38), written in the weak form (3.39), has a unique solution*

$$p \in H^1(0, T; H^1(\Omega)^*) \cap C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

PROOF. The proof is analogous to the first part of the proof of Proposition 3.2. In fact, one can again devise a Faedo-Galerkin approximation scheme, for which estimates similar to the ones leading to (3.14) can be performed. An estimate resembling (3.21) cannot be derived since $\beta_1 (\bar{\varphi}(T) - \varphi_\Omega)$ does not necessarily belong to $H^1(\Omega)$. One then obtains a weak solution that enjoys the asserted regularity and turns out to be unique. Since these arguments are rather standard and straightforward, we can afford to omit the details here. ■

We are now in the position to eliminate ξ^h from (3.35). We have the following result.

Theorem 3.6. *Assume that the hypotheses (H1)–(H8) are fulfilled, and let $\bar{\mathbf{v}} \in \mathcal{V}_{\text{ad}}$ be an optimal control for problem (CP) with associated state $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$ and adjoint state p . Then we have for every $\mathbf{v} \in \mathcal{V}_{\text{ad}}$ the inequality*

$$\beta_3 \int_0^T \int_\Omega \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) \, dx \, dt + \int_0^T \int_\Omega p(\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla \bar{\varphi} \, dx \, dt \geq 0. \quad (3.40)$$

PROOF. This is a standard calculation that can be left to the reader. We only note that we have

$$\begin{aligned} & \beta_1 \int_0^T \int_\Omega (\bar{\varphi} - \varphi_Q) \xi^h \, dx \, dt + \beta_2 \int_\Omega (\bar{\varphi}(T) - \varphi_\Omega) \xi^h(T) \, dx \\ &= \beta_1 \int_0^T \int_\Omega (\bar{\varphi} - \varphi_Q) \xi^h \, dx \, dt + \int_0^T (\langle p_t(t), \xi^h(t) \rangle + \langle \xi_t^h(t), p(t) \rangle) \, dt \\ &= \int_0^T \int_\Omega p(\mathbf{v} - \bar{\mathbf{v}}) \cdot \nabla \bar{\varphi} \, dx \, dt, \end{aligned} \quad (3.41)$$

where the last equality easily follows from inserting for $p_t(t)$ and $\xi_t^h(t)$, using the adjoint system and the linearized system, and then integrating by parts. ■

Remark 5. The state system (1.2)–(1.6), written for $\varphi = \bar{\varphi}$, the adjoint system and the variational inequality (3.40) form together the first-order necessary optimality conditions. Moreover, since \mathcal{V}_{ad} is a nonempty, closed, and convex subset of $L^2(Q)^3$, (3.40) implies that for $\beta_3 > 0$ the optimal control $\bar{\mathbf{v}}$ is the $L^2(Q)^3$ -orthogonal projection of $-\beta_3^{-1} p \nabla \bar{\varphi}$ onto \mathcal{V}_{ad} .

References

- [1] P. W. BATES AND J. HAN, *The Neumann boundary problem for a nonlocal Cahn–Hilliard equation*, J. Differential Equations, **212** (2005), pp. 235–277.

- [2] L. BLANK, M. BUTZ, H. GARCKE, L. SARBU, AND V. STYLES, *Allen–Cahn and Cahn–Hilliard variational inequalities solved with optimization techniques*, in: *Constrained Optimization and Optimal Control for Partial Differential Equations*, vol. 160 of *Internat. Ser. Numer. Math.*, Birkhäuser/Springer Basel AG, Basel, 2012, pp. 21–35.
- [3] J. W. CAHN, *On spinodal decomposition*, *Acta Metall.*, **9** (1961), pp. 795–801.
- [4] J. W. CAHN AND J. E. HILLIARD, *Free energy of a nonuniform system. I. Interfacial free energy*, *J. Chem. Phys.*, **28** (1958), pp. 258–267.
- [5] J. W. CAHN AND J. E. HILLIARD, *Spinodal decomposition: A reprise*, *Acta Metall.*, **19** (1971), pp. 151–161.
- [6] P. COLLI, S. FRIGERI, AND M. GRASSELLI, *Global existence of weak solutions to a nonlocal Cahn–Hilliard–Navier–Stokes system*, *J. Math. Anal. Appl.*, **386** (2012), pp. 428–444.
- [7] P. COLLI, G. GILARDI, P. PODIO-GUIDUGLI, AND J. SPREKELS, *Distributed optimal control of a nonstandard system of phase field equations*, *Contin. Mech. Thermodyn.*, **24** (2012), no. 4–6, pp. 437–459.
- [8] P. COLLI, G. GILARDI, AND J. SPREKELS, *Analysis and optimal boundary control of a nonstandard system of phase field equations*, *Milan J. Math.*, **80** (2012), no. 1, pp. 119–149.
- [9] P. COLLI, P. KREJČÍ, E. ROCCA, AND J. SPREKELS, *Nonlinear evolution inclusions arising from phase change models*, *Czechoslovak Math. J.*, **57** (2007), pp. 1067–1098.
- [10] P.-É. DRUET, O. KLEIN, J. SPREKELS, F. TRÖLTZSCH, AND I. YOUSEPT, *Optimal control of three-dimensional state-constrained induction heating problems with nonlocal radiation effects*, *SIAM J. Control Optim.*, **49** (2011), no. 4, pp. 1707–1736.
- [11] C. M. ELLIOTT AND H. GARCKE, *On the Cahn–Hilliard equation with degenerate mobility*, *SIAM J. Math. Anal.*, **27** (1996), pp. 404–423.
- [12] H. EMMERICH, *The Diffuse Interface Approach in Materials Science*, Springer, Berlin Heidelberg, 2003.
- [13] S. FRIGERI, C. G. GAL, AND M. GRASSELLI, *On nonlocal Cahn–Hilliard–Navier–Stokes systems in two dimensions*, submitted.
- [14] S. FRIGERI, M. GRASSELLI, AND E. ROCCA, *A diffuse interface model for two-phase incompressible flows with nonlocal interactions and nonconstant mobility*, preprint arXiv:1303.6446, 2013, 45 pages.
- [15] H. GAJEWSKI AND K. ZACHARIAS, *On a nonlocal phase separation model*, *J. Math. Anal. Appl.*, **286** (2003), no. 1, pp. 11–31.
- [16] C. G. GAL AND M. GRASSELLI, *Longtime behavior of nonlocal Cahn–Hilliard equations*, *Discrete Contin. Dyn. Syst. Ser. A*, to appear.

- [17] G. GIACOMIN AND J. L. LEBOWITZ, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys., **87** (1997), pp. 37–61.
- [18] G. GIACOMIN AND J. L. LEBOWITZ, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math., **58** (1998), pp. 1707–1729.
- [19] M. HINTERMÜLLER AND D. WEGNER, *Distributed optimal control of the Cahn–Hilliard system including the case of a double-obstacle homogeneous free energy density*, SIAM J. Control Optim., **50** (2012), no. 1, pp. 388–418.
- [20] M. HINTERMÜLLER, M. HINZE AND M. H. TBER, *An adaptive finite-element Moreau–Yosida-based solver for a non-smooth Cahn–Hilliard problem*, Optim. Methods Softw., **26** (2011), no. 4–5, pp. 777–811.
- [21] P. KREJČÍ, E. ROCCA, AND J. SPREKELS, *A nonlocal phase-field model with nonconstant specific heat*, Interfaces Free Bound., **9** (2007), pp. 285–306.
- [22] P. KREJČÍ, E. ROCCA, AND J. SPREKELS, *Non-local temperature dependent phase-field model for non-isothermal phase transitions*, J. London Math. Soc., **76** (2007), pp. 197–210.
- [23] J.-L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
- [24] S.-O. LONDEN AND H. PETZELTOVÁ, *Convergence of solutions of a non-local phase-field system*, Discrete Contin. Dyn. Syst. Ser. S, **4** (2011), pp. 653–670.
- [25] S.-O. LONDEN AND H. PETZELTOVÁ, *Regularity and separation from potential barriers for a non-local phase-field system*, J. Math. Anal. Appl., **379** (2011), no. 2, pp. 724–735.
- [26] C. MEYER AND I. YOUSEPT, *State-constrained optimal control of semilinear elliptic equations with nonlocal radiation interface conditions*, SIAM J. Control Optim., **48** (2009), no. 2, pp. 734–755.
- [27] C. MEYER, P. PHILIP, AND F. TRÖLTZSCH, *Optimal control of a semilinear PDE with nonlocal radiation interface conditions*, SIAM J. Control Optim., **45** (2006), no. 2, pp. 699–721.
- [28] P. PHILIP, *Analysis, optimal control, and simulation of conductive-radiative heat transfer*, Ann. Acad. Rom. Sci. Ser. Math. Appl., **2** (2010), no. 2, pp. 171–204.
- [29] J. S. ROWLINSON, Translation of J. D. van der Waals, *The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density*, J. Statist. Phys., **20** (1979), pp. 197–244.
- [30] J. SIMON, *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pura Appl. (4), **146** (1987), pp. 65–96.
- [31] F. TRÖLTZSCH, *Optimal Control of Partial Differential Equations. Theory, Methods and Applications*, Graduate Studies in Mathematics, 112, American Mathematical Society, Providence, RI, 2010.

- [32] Q.-F. WANG AND S.-I. NAKAGIRI, *Weak solutions of Cahn–Hilliard equations having forcing terms and optimal control problems*, Mathematical models in functional equations (Japanese) (Kyoto, 1999), Sūrikaiseikikenkyūsho Kōkyūroku No. 1128 (2000), pp. 172–180.
- [33] X. ZHAO AND C. LIU, *Optimal control of the convective Cahn–Hilliard equation*, Appl. Anal., **92** (2013), no. 5, pp. 1028–1045.