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ON OVERGROUPS OF DISTINGUISHED UNIPOTENT ELEMENTS IN REDUCTIVE GROUPS AND FINITE GROUPS OF LIE TYPE

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Dedicated to the fond memory of Gary Seitz

Abstract. Suppose G is a simple algebraic group defined over an algebraically closed field of good characteristic p. In 2018 Korhonen showed that if H is a connected reductive subgroup of G which contains a distinguished unipotent element u of G of order p, then H is G-irreducible in the sense of Serre. We present a short and uniform proof of this result using so-called *good* A_1 subgroups of G, introduced by Seitz. We also formulate a counterpart of Korhonen's theorem for overgroups of u which are finite groups of Lie type. Moreover, we generalize both results above by removing the restriction on the order of u under a mild condition on p depending on the rank of G, and we present an analogue of Korhonen's theorem for Lie algebras.

1. Introduction and main results

Throughout, G is a connected reductive linear algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$ and H is a closed subgroup of G.

Following Serre [30], we say that H is *G-completely reducible* (G-cr for short) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P, and that H is *G-irreducible* (G-ir for short) provided H is not contained in any proper parabolic subgroup of G at all. Clearly, if H is G-irreducible, it is trivially G-completely reducible, and an overgroup of a G-irreducible subgroup is again G-irreducible; for an overview of this concept see [6], [29] and [30]. Note that in case $G = \mathrm{GL}(V)$ a subgroup H is G-completely reducible exactly when V is a semisimple H-module and it is G-irreducible precisely when V is an irreducible H-module. Recall that if H is G-completely reducible, then the identity component H° of H is reductive, [30, Prop. 4.1].

A unipotent element u of G is *distinguished* provided any torus in the centraliser $C_G(u)$ of u in G is central in G. Likewise, a nilpotent element e of the Lie algebra $\mathrm{Lie}(G)$ of G is *distinguished* provided any torus in the centraliser $C_G(e)$ of e in G is central in G, see [9, §5.9] and [15, §4.1]. For instance, regular unipotent elements in G are distinguished, and so are regular nilpotent elements in $\mathrm{Lie}(G)$ [32, III 1.14] (or [9, Prop. 5.1.5]). Overgroups of regular unipotent elements have attracted much attention in the literature, e.g., see [35], [27], [37], [20], and [7].

In [16], Korhonen proves the following remarkable result in the special case when G is simple.

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Theorem 1.1 ([16, Thm. 6.5]). *Suppose G is connected reductive and p is good for G. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p. Then H is G-irreducible.*

Korhonen's proof of Theorem 1.1 depends on checks for the various possible Dynkin types for simple G. E.g., for G simple of exceptional type, Korhonen's argument relies on long exhaustive case-by-case investigations from [18], where all connected reductive non-G-cr subgroups are classified in the exceptional type groups in good characteristic. For classical G, Korhonen requires an intricate classification of all $\mathrm{SL}_2(k)$-representations on which a non-trivial unipotent element of $\mathrm{SL}_2(k)$ acts with at most one Jordan block of size p. Our main aim is to give a short uniform proof of Theorem 1.1 in §5 without resorting to further case-by-case checks, using a landmark result by Seitz (see §4.2).

Remark 1.2. Suppose as in Theorem 1.1, that G is simple classical with natural module V, and $p \geq \dim V > 2$. Then, thanks to [14, Prop. 3.2], V is semisimple as an H°-module, and by [30, (3.2.2(b))], this is equivalent to H° being G-cr. Then H is G-ir, by Lemma 3.2. This gives a short uniform proof of the conclusion of Theorem 1.1 in this case, as the bound $p \geq \dim V > 2$ ensures that every distinguished unipotent element (including the regular ones) is of order p. The conclusion can fail if the bound is not satisfied: see Theorem 1.3.

There are only a few cases when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p, by work of Proud–Saxl–Testerman [26, Lem. 4.1, Lem. 4.2] (see Lemmas 4.1 and 4.3). In this case the conclusion of Theorem 1.1 fails precisely in one instance, as observed in [16, Prop. 1.2] (Example 4.2), else it is valid (Example 4.4). Combining the cases when p is bad for G with Theorem 1.1, we recover Korhonen's main theorem from [16]. Here and later on, we say that a subgroup of G is of *type* A_1 if it is isomorphic to $\mathrm{SL}_2(k)$ or $\mathrm{PSL}_2(k)$.

Theorem 1.3 ([16, Thm. 1.3]). *Let G be simple and let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G of order p. Then H is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

Our second goal is an extension of Theorem 1.1 to finite groups of Lie type in G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G, so that the finite fixed point subgroup $G_\sigma = G(q)$ is a finite group of Lie type over the field $\mathbb{F}_q$ of q elements. For a Steinberg endomorphism σ of G and a connected reductive σ-stable subgroup H of G, σ is also a Steinberg endomorphism for H with finite fixed point subgroup $H_\sigma = H \cap G_\sigma$, [33, 7.1(b)]. Obviously, one cannot directly appeal to Theorem 1.1 to deduce anything about H_σ, because H_σ° is trivial. For the notion of a q-Frobenius endomorphism, see §2.4.

Theorem 1.4. *Let $H \subseteq G$ be connected reductive groups. Suppose p is good for G. Let $\sigma\colon G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G admits components of exceptional type, then assume $q > 7$. If H_σ contains a distinguished unipotent element of G of order p, then H_σ is G-irreducible.*

Combining Theorem 1.4 with the aforementioned results from [26], we are able to deduce the following analogue of Theorem 1.3 for finite subgroups of Lie type in G.

Theorem 1.5. *Let G be simple and let H be a connected reductive subgroup of G. Let $\sigma\colon G \to G$ be a Steinberg endomorphism stabilizing H such that $\sigma|_H$ is a q-Frobenius endomorphism of H. If G is of exceptional type, then assume $q > 7$. If H_σ contains a distinguished* *unipotent element of G of order p, then H_σ is G-irreducible, unless $p = 2$, G is of type C_2, and H is a type A_1 subgroup of G.*

In the special instance in Theorems 1.4 and 1.5 when H_σ contains a regular unipotent element u from G, the conclusion of both theorems holds without any restriction on the order of u and without any restriction on q (and without any exceptions of the type seen in Theorem 1.5); see [7, Thm. 1.3].

Section 2 contains background material. In Section 3 we extend Theorem 1.1 to distinguished unipotent elements of arbitrary order under an extra hypothesis on p, and we prove an analogue of Theorem 1.1 for Lie algebras under the same hypothesis. We recall Seitz's notion of good A_1 subgroups in Section 4. Theorems 1.1 and 1.3–1.5 are proved in Section 5.

2. Preliminaries

2.1. **Notation.** Throughout, we work over an algebraically closed field k of characteristic $p \geq 0$. All affine varieties are considered over k and are identified with their k-points. A linear algebraic group H over k has identity component H°; if $H = H^\circ$, then we say that H is *connected*. We denote by $R_u(H)$ the *unipotent radical* of H; if $R_u(H)$ is trivial, then we say H is *reductive*.

Throughout, G denotes a connected reductive linear algebraic group over k. All subgroups of G considered are closed. By $\mathscr{D}G$ we denote the derived subgroup of G, and likewise for subgroups of G. We denote the Lie algebra of G by $\mathrm{Lie}(G)$ or by $\mathfrak{g}$.

Let $Y(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ denote the set of cocharacters of G. For $\mu \in Y(G)$ and $g \in G$ we define the *conjugate cocharacter* $g \cdot \mu \in Y(G)$ by $(g \cdot \mu)(t) = g\mu(t)g^{-1}$ for $t \in \mathbb{G}_m$; this gives a left action of G on $Y(G)$. For H a subgroup of G, let $Y(H) := Y(H^\circ) = \mathrm{Hom}(\mathbb{G}_m, H)$ denote the set of cocharacters of H. There is an obvious inclusion $Y(H) \subseteq Y(G)$.

Fix a Borel subgroup B of G containing a maximal torus T. Let $\Phi = \Phi(G,T)$ be the root system of G with respect to T, let $\Phi^+ = \Phi(B,T)$ be the set of positive roots of G, and let $\Sigma = \Sigma(G,T)$ be the set of simple roots of Φ^+. For each $\alpha \in \Phi$ we have a root subgroup U_α of G. For α in Φ, let $x_\alpha : \mathbb{G}_a \to U_\alpha$ be a parametrization of the root subgroup U_α of G.

2.2. **Good primes.** A prime p is said to be *good* for G if it does not divide any coefficient of any positive root when expressed as a linear combination of simple ones. Else p is called *bad* for G, [32, §4]. Explicitly, if G is simple, p is *good* for G provided $p > 2$ in case G is of Dynkin type B_n, C_n, or D_n; $p > 3$ in case G is of Dynkin type E_6, E_7, F_4 or G_2 and $p > 5$ in case G is of type E_8. We observe that if L is a Levi subgroup of G and p is good for G, then it is also good for L.

2.3. **Springer isomorphisms.** If p is good for G, then there exists a G-equivariant homeomorphism $\phi : \mathscr{U} \to \mathscr{N}$ between the unipotent variety $\mathscr{U}$ of G and the nilpotent cone $\mathscr{N}$ of $\mathrm{Lie}(G)$. Such a map is called a *Springer isomorphism*, see [32, III, 3.12] and [4, Cor. 9.3.4]. Its inverse is the analogue of exponentiation in characteristic 0. By means of such a map, all the results below formulated for unipotent elements in G stem from analogues for nilpotent elements in $\mathrm{Lie}(G)$. In what follows we fix such a Springer isomorphism once and for all.

2.4. **Steinberg endomorphisms of G.** Recall that a *Steinberg endomorphism* of G is a surjective homomorphism $\sigma : G \to G$ such that the corresponding fixed point subgroup $G_\sigma := \{g \in G \mid \sigma(g) = g\}$ of G is finite. Frobenius endomorphisms σ_q of reductive groups over finite fields are familiar examples, giving rise to *finite groups of Lie type* $G(q)$. See Steinberg [33] for a detailed discussion (for this terminology, see [12, Def. 1.15.1b]). The set of all Steinberg endomorphisms of G is a subset of the set of all isogenies $G \to G$ (see [33, 7.1(a)]) which encompasses in particular all (generalized) Frobenius endomorphisms, i.e., endomorphisms of G some power of which are Frobenius endomorphisms corresponding to some $\mathbb{F}_q$-rational structure on G. In that case we also denote the finite group of Lie type G_σ by $G(q)$. If $\mathscr{S}$ is a σ-stable set of closed subgroups of G, then $\mathscr{S}_\sigma$ denotes the subset consisting of all σ-stable members of $\mathscr{S}$.

If $\sigma_q : G \to G$ is a standard q-power Frobenius endomorphism of G, then there exists a σ_q-stable maximal torus T and Borel subgroup $B \supseteq T$, and with respect to a chosen parametrisation of the root groups as above, we have $\sigma_q(x_\alpha(t)) = x_\alpha(t^q)$ for each $\alpha \in \Phi$ and $t \in \mathbb{G}_a$, cf. [12, Thm. 1.15.4(a)]. Following [26], we call a generalized Frobenius endomorphism σ a *q-Frobenius endomorphism* provided $\sigma = \tau\sigma_q$, where τ is an algebraic automorphism of G of finite order, σ_q is a standard q-power Frobenius endomorphism of G, and σ_q and τ commute. When p is bad for G, a q-Frobenius endomorphism does not involve a twisted Steinberg endomorphism, cf. [26, §3]. In particular, if G is simple and p is good for G, then any Steinberg endomorphism of G is a q-Frobenius endomorphism, cf. [33, §11]. If G is not simple and p is bad for G, then a generalized Frobenius map may fail to factor into a field and algebraic automorphism of G, cf. [13, Ex. 1.3].

2.5. **Bala–Carter Theory.** We recall some relevant results and concepts from Bala–Carter Theory. A parabolic subgroup P of G admits a dense open orbit on its unipotent radical $R_u(P)$, the so-called *Richardson orbit*; see [9, Thm. 5.2.1]. A parabolic subgroup P of G is called *distinguished* provided $\dim(\mathscr{D}P/R_u(P)) = \dim(R_u(P)/\mathscr{D}R_u(P))$, cf. [9, §5.8]. For G simple, the distinguished parabolic subgroups of G (up to G-conjugacy) were worked out in [2] and [3]; see [9, pp. 174–177].

The following is the celebrated Bala–Carter theorem, see [9, Thm. 5.9.5, Thm. 5.9.6], which is valid in good characteristic, thanks to work of Pommerening [23], [24]. For the Lie algebra versions see also [15, Prop. 4.7, Thm. 4.13].

Theorem 2.1. *Suppose p is good for G.*

- (i) *There is a bijective map between the G-conjugacy classes of distinguished unipotent elements of G and conjugacy classes of distinguished parabolic subgroups of G. The unipotent class corresponding to a given parabolic subgroup P contains the dense P-orbit on $R_u(P)$.*
- (ii) *There is a bijective map between the G-conjugacy classes of unipotent elements of G and conjugacy classes of pairs (L, P), where L is a Levi subgroup of G and P is a distinguished parabolic subgroup of $\mathscr{D}L$. The unipotent class corresponding to the pair (L, P) contains the dense P-orbit on $R_u(P)$.*

Remark 2.2. (i). Let $u \in G$ be unipotent. Let S be a maximal torus of $C_G(u)$. Then u is distinguished in the Levi subgroup $C_G(S)$ of G, for S is the unique maximal torus of $C_{C_G(S)}(u)$. Conversely, if L is a Levi subgroup of G with u distinguished in L, then the connected center of L is a maximal torus of $C_G(u)^\circ$, see [15, Rem. 4.7].

(ii). Let $\sigma : G \to G$ be a Steinberg endomorphism of G and let $u \in G_\sigma$ be unipotent. Then $C_G(u)^\circ$ is σ-stable. The set of all maximal tori of $C_G(u)^\circ$ is σ-stable and $C_G(u)^\circ$ is transitive on that set ([31, Thm. 6.4.1]). Thus the Lang–Steinberg Theorem ([32, I 2.7]) provides a σ-stable maximal torus, say S, of $C_G(u)^\circ$. Then, by part (i), $L = C_G(S)$ is a σ-stable Levi subgroup of G and u is distinguished in L. Moreover, since the centralizer of S in $C_G(u)^\circ$ is connected, by [31, Thm. 6.4.7(i)], it follows from [32, Cor. I 2.8(a)] that $(C_G(u)^\circ)_\sigma$ is transitive on the set of all σ-stable maximal tori of $C_G(u)^\circ$ and thus $(C_G(u)^\circ)_\sigma$ is transitive on the set of all σ-stable Levi subgroups $L = C_G(S)$ of G, where S is as above, and thus so is $C_{G_\sigma}(u)$.

2.6. **Cocharacters and parabolic subgroups of G.** Let $\lambda \in Y(G)$. Without loss, we may assume that $\lambda \in Y(T)$. Recall that λ affords a $\mathbb{Z}$-grading on $\mathrm{Lie}(G) = \mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j,\lambda)$, where $\mathfrak{g}(j,\lambda) := \{e \in \mathfrak{g} \mid \mathrm{Ad}(\lambda(t))e = t^j e \text{ for every } t \in \mathbb{G}_m\}$ is the j-weight space of $\mathrm{Ad}(\lambda(\mathbb{G}_m))$ on $\mathfrak{g}$, cf. [9, §5.5] or [15, §5.1]. Let $\mathfrak{p}_\lambda := \bigoplus_{j\geq 0} \mathfrak{g}(j,\lambda)$. Then there is a unique parabolic subgroup P_λ with $\mathrm{Lie}\, P_\lambda = \mathfrak{p}_\lambda$ and $C_G(\lambda) := C_G(\lambda(\mathbb{G}_m))$ is a Levi subgroup of P_λ. Specifically, we have $U_\alpha \subseteq P_\lambda$ if and only if $\langle \lambda, \alpha\rangle \geq 0$, where $\langle\ ,\ \rangle : Y(T) \times X(T) \to \mathbb{Z}$ is the usual pairing between cocharacters and characters $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of T. We have $U_\alpha \subseteq C_G(\lambda)$ if and only if $\langle \lambda, \alpha\rangle = 0$, and $R_u(P_\lambda)$ is generated by the U_α with $\langle \lambda, \alpha\rangle > 0$; cf. the proof of [31, Prop. 8.4.5].

Set $J := \{\alpha \in \Sigma \mid \langle \alpha, \lambda\rangle = 0\}$. Then $P_\lambda = P_J = \big\langle T, U_\alpha \mid \langle \alpha, \lambda\rangle \geq 0\big\rangle$ is the *standard parabolic subgroup* of G associated with $J \subseteq \Sigma$.

Let $\rho = \sum_{\alpha\in\Sigma} c_{\alpha\rho}\alpha$ be the highest root in Φ^+. Define $\mathrm{ht}_J(\rho) := \sum_{\alpha\in\Sigma\setminus J} c_{\alpha\rho}$. In view of Theorem 2.1, the following gives the order of a distinguished unipotent element in good characteristic.

Lemma 2.3 ([36, Order Formula 0.4])**.** *Let p be good for G. Let $P = P_J$ be a distinguished parabolic subgroup of G and let u be in the Richardson orbit of P on $R_u(P)$. Then the order of u is* $\min\{p^a \mid p^a > \mathrm{ht}_J(\rho)\}$.

2.7. **Cocharacters associated to nilpotent and unipotent elements.** The Jacobson–Morozov Theorem allows one to associate an $\mathfrak{sl}(2)$-triple to any given non-zero nilpotent element in $\mathscr{N}$ in characteristic zero or large positive characteristic. This is an indispensable tool in the Dynkin–Kostant classification of the nilpotent orbits in characteristic zero as well as in the Bala–Carter classification of unipotent conjugacy classes of G in large prime characteristic, see [9, §5.9]. In good characteristic there is a replacement for $\mathfrak{sl}(2)$-triples, so-called *associated cocharacters*; see Definition 2.4 below. These cocharacters are important tools in the classification theory of unipotent and nilpotent classes of reductive algebraic groups in good characteristic, see for instance [15, §5] and [25]. We recall the relevant concept of cocharacters associated to a nilpotent element following [15, §5.3].

Definition 2.4. A cocharacter $\lambda \in Y(G)$ of G is *associated* to $e \in \mathscr{N}$ provided $e \in \mathfrak{g}(2,\lambda)$ and there exists a Levi subgroup L of G such that e is distinguished nilpotent in $\mathrm{Lie}\, L$ and $\lambda(\mathbb{G}_m) \leq \mathscr{D}L$.

A cocharacter $\lambda \in Y(G)$ of G is *associated* to $u \in \mathscr{U}$ provided it is associated to $\phi(u)$, where $\phi : \mathscr{U} \to \mathscr{N}$ is a fixed Springer isomorphism as in §2.3; cf. [22, Rem. 23]. Following [11, Def. 2.13], we write

$$\Omega^a_G(u) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } u\}$$

for the set of cocharacters of G associated to u. This notation stems from the fact that associated cocharacters are destabilising cocharacters of G for u in the sense of Kempf–Rousseau theory, see [25] and [22]. Likewise, for M a connected reductive subgroup of G containing u, we write $\Omega^a_M(u)$ for the set of cocharacters of M that are associated to u.

Remark 2.5. Let $u \in G$ be unipotent, $\lambda \in \Omega^a_G(u)$, and $g \in C_G(u)$. Then $g \cdot \lambda$ is also associated to u, cf. [15, §5.3]. Proposition 2.6(ii) gives a converse to this property.

We require some basic facts about cocharacters associated to unipotent elements. The following results are [15, Lem. 5.3; Prop. 5.9] for nilpotent elements, see also [25, Thm. 2.3, Prop. 2.5].

Proposition 2.6. *Suppose p is good for G. Let $u \in G$ be unipotent.*

(i) *$\Omega^a_G(u) \neq \varnothing$, i.e., cocharacters of G associated to u exist.*
(ii) *$C_G(u)^\circ$ acts transitively on $\Omega^a_G(u)$.*
(iii) *Let $\lambda \in \Omega^a_G(u)$ and let P_λ be the parabolic subgroup of G defined by λ as in §2.6. Then P_λ only depends on u and not on the choice of λ.*
(iv) *Let $\lambda \in \Omega^a_G(u)$ and let $P(u) := P_\lambda$ be as in (iii). Then $C_G(u) \subseteq P(u)$.*

If u is distinguished in G, then the parabolic subgroup $P(u)$ of G from Proposition 2.6(iii) is a distinguished parabolic subgroup of G and u belongs to the Richardson orbit of $P(u)$ on its unipotent radical, see Theorem 2.1(i); cf. [22, Prop. 22].

Let u be unipotent in G and let $\lambda \in \Omega^a_G(u)$. Let $P = P(u)$ be the canonical parabolic subgroup defined by u from Proposition 2.6. Then $P = C_G(\lambda)R_u(P)$ is a Levi decomposition of P. Following [15, §5.10] and [25, §2.4], we define the subgroup

$$C_G(u, \lambda) := C_G(u) \cap C_G(\lambda) \qquad (2.7)$$

of $C_G(u)$. In view of [25, Prop. 2.5], our next result is [25, Thm. 2.3(iii)], see also [15, Prop. 5.10, Prop. 5.11].

Proposition 2.8. *Suppose p is good for G. Let $u \in G$ be unipotent and let $\lambda \in \Omega^a_G(u)$. Then $C_G(u)$ is the semidirect product of $C_G(u, \lambda)$ and $R_u(C_G(u))$, and $C_G(u, \lambda)$ is reductive.*

We note that cocharacters of G associated to a unipotent element u of G are compatible with certain group-theoretic operations; see [15, §5.6].

Remark 2.9. Suppose $p > 0$ and $u \in G$ is unipotent of order p contained in a subgroup A of G of type A_1. Such a subgroup A always exists when p is good, and when p is bad there is essentially only one exception, due to Testerman [36] and Proud–Saxl–Testerman [26] — see Theorems 4.5 and 4.6 below. Then, since p is good for A, by Proposition 2.6(i) there exists a cocharacter $\lambda \in \Omega^a_A(u)$. Note that $\lambda(\mathbb{G}_m)$ is a maximal torus in A.

It follows from the work of Pommerening [23], [24] that the description of the unipotent classes in characteristic 0 is identical to the one for G when p is good for G. In both instances these are described by so-called *weighted Dynkin diagrams*. As a result, an associated cocharacter to a unipotent element acts with the very same weights on the Lie algebra of G as its counterpart does in characteristic 0. This fact is used in the proof of the following result by Lawther [17, Thm. 1]; cf. the proof of [28, Prop. 4.2].

Lemma 2.10. *Let G be simple and suppose $u \in G$ is unipotent. Suppose p is good for G. Let $\lambda \in \Omega_G^a(u)$. Denote by ω_G the highest weight of $\lambda(\mathbb{G}_m)$ on* $\mathrm{Lie}(G)$. *Then u has order p if and only if $\omega_G \leq 2p - 2$.*

The concept of associated cocharacter is not only a convenient replacement for $\mathfrak{sl}(2)$-triples from the Jacobson–Morozov Theory, it is a very powerful tool in the classification theory. Specifically, in [25] Premet showcases a conceptual and uniform proof of Pommerening's extension of the Bala–Carter Theorem 2.1 to good characteristic. His proof uses the fact that associated cocharacters are *optimal* in the geometric invariant theory sense of Kempf–Rousseau–Hesselink. In §5, we demonstrate the utility of this concept further with short uniform proofs of parts of Seitz's Theorem 4.10 on good A_1 subgroups.

2.8. **Cocharacters associated to distinguished elements.** The linchpin of our proofs of Theorems 1.1 and 1.4 is the following fact.

Lemma 2.11 ([11, Lem. 3.1])**.** *Suppose p is good for G. Let $M \subset G$ be a connected reductive subgroup of G. Let $u \in M$ be a distinguished unipotent element of G. Then $\Omega_M^a(u) = \Omega_G^a(u) \cap Y(M)$.*

The assertion of the lemma fails in general if u is not distinguished in G, even when p is good for both M and G, e.g., see [15, Rem. 5.12]. However, we do have the following result for all unipotent elements in good characteristic.

Lemma 2.12 ([11, Cor. 3.22])**.** *Suppose p is good for G. Let $L \subset G$ be a Levi subgroup of G. Let $u \in L$ be unipotent. Then $\Omega_L^a(u) = \Omega_G^a(u) \cap Y(L)$.*

3. Generalizations of Theorems 1.1 and 1.4

The aim of this section is to generalize Theorem 1.1 to the situation where there is no restriction on the order of the unipotent element at hand, see Theorem 3.1. For a unipotent element u in G to be distinguished is a mere condition on the structure of the centralizer $C_G(u)$ of u in G. The extra condition for u to have order p is thus somewhat artificial. The restriction in Theorem 1.1 is due to the methods used in [16] and in our proofs in §5, which require the unipotent element to lie in a subgroup of type A_1; such an element must obviously have order p.

Along the way we prove an analogue of Theorem 1.1 for Lie algebras, i.e., under the hypothesis that $\mathrm{Lie}(H)$ contains a distinguished nilpotent element of $\mathrm{Lie}(G)$. In order to state this theorem, we need to introduce an invariant $a(G)$ of G from [30, §5.2]: for G simple, set $a(G) = \mathrm{rk}(G) + 1$, where $\mathrm{rk}(G)$ is the rank of G. For reductive G, let $a(G) = \max\{1, a(G_1), \ldots, a(G_r)\}$, where $G_1, \ldots, G_r$ are the simple components of G.

Theorem 3.1. *Suppose $p \geq a(G)$. Let H be a reductive subgroup of G. Suppose H° contains a distinguished unipotent element of G or* $\mathrm{Lie}(H)$ *contains a distinguished nilpotent element of* $\mathrm{Lie}(G)$. *Then H is G-irreducible.*

The following analogue of [7, Cor. 4.6] shows that in order to derive the G-irreducibility of H in Theorem 3.1, it suffices to show that H is G-cr, cf. [16, Lem. 6.1]. This also applies to Theorem 1.1 and Theorem 1.4.

Lemma 3.2. *Let H be a G-completely reducible subgroup of G. Suppose that H contains a distinguished unipotent element u of G or* $\mathrm{Lie}(H)$ *contains a distinguished nilpotent element e of* $\mathrm{Lie}(G)$. *Then H is G-irreducible.*

Proof. Suppose H is contained in a parabolic subgroup P of G. Then, by hypothesis, H is contained in a Levi subgroup L of P. As the latter is the centraliser of a torus S in G, S centralises u (resp., e) and so S is central in G. Hence $L = G$, which implies $P = G$. □

Along with Lemma 3.2, the following theorem of Serre immediately yields Theorem 3.1.

Theorem 3.3 ([30, Thm. 4.4])**.** *Suppose $p \geq a(G)$ and $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Proof of Theorem 3.1. Since $p \geq a(G)$, Theorem 3.3 applied to H° shows the latter is G-cr. Thus H° is G-ir by Lemma 3.2, and so is H. □

Remarks 3.4. (i). The characteristic restriction in Theorem 3.1 (and Theorem 3.3) is needed; indeed the bound $p \geq a(G)$ is sharp (see Theorem 1.3).

(ii). The condition in Theorem 3.1 requiring H° to contain a distinguished unipotent element of G is also necessary, as for instance the finite unipotent subgroup of G generated by a given distinguished unipotent element of G is not G-cr [30, Prop. 4.1].

(iii). Under the given hypotheses, Theorem 3.1 applies to an arbitrary distinguished unipotent element of G, irrespective of its order. For Theorem 1.1 to achieve the same uniform result, p has to be sufficiently large to guarantee that the chosen element has order p. For G simple classical with natural module V, this requires the bound $p \geq \dim V$; see Remark 1.2. For G simple of exceptional type, this requires the following bounds: $p > 11$ for E_6, $p > 17$ for E_7, $p > 29$ for E_8, $p > 11$ for F_4, and $p > 5$ for G_2; see [36, Prop. 2.2]. So in many cases the bound $p \geq a(G)$ from Theorem 3.1 is better.

(iv). For an instance when p is bad for G so that Theorem 1.1 does not apply, but Theorem 3.1 does, see Example 4.4.

(v). Theorem 3.1 generalizes [7, Thm. 3.2] which consists of the analogue in the special instance when the distinguished element is regular in G (or $\mathrm{Lie}(G)$). Note that in this case no restriction on p is needed, cf. [37, Thm. 1.2], [20, Thm. 1], [7, Thm. 3.2].

(vi). In characteristic 0, a subgroup H of G is G-cr if and only if it is reductive, [30, Prop. 4.1]. So in that case the conclusion of Theorem 3.1 follows directly from Lemma 3.2.

Once again, in the presence of a Steinberg endomorphism σ of G, one cannot appeal to Theorem 3.1 directly to deduce anything about H_σ, because H_σ° is trivial. In Corollary 3.6 we present an analogue of Theorem 3.1 for the finite groups of Lie type H_σ under an additional condition stemming from [5].

Note that for S a torus in G, we have $C_G(S) = C_G(s)$ for some $s \in S$, see [8, III Prop. 8.18].

Proposition 3.5 ([5, Prop. 3.2])**.** *Let $H \subseteq G$ be connected reductive groups. Let $\sigma \colon G \to G$ be a Steinberg endomorphism that stabilises H and a maximal torus T of H. Suppose*

(i) *$C_G(T) = C_G(t)$, for some $t \in T_\sigma$, and*

(ii) *H_σ meets every T-root subgroup of H non-trivially.*

Then H_σ and H belong to the same parabolic and the same Levi subgroups of G. In particular, H is G-completely reducible if and only if H_σ is G-completely reducible; similarly, H is G-irreducible if and only if H_σ is G-irreducible.

Without condition (i), the proposition is false in general, see [5, Ex. 3.2]. The following is an immediate consequence of Theorem 3.1 and Proposition 3.5.

Corollary 3.6. *Suppose G, H and σ satisfy the hypotheses of Proposition 3.5. Suppose in addition that $p \geq a(G)$. If H_σ contains a distinguished unipotent element of G, then H_σ is G-irreducible.*

Corollary 3.6 generalizes [7, Thm. 1.3] which consists of the analogue in the special instance when the distinguished element is regular in G. Note in this case no restriction on p is needed.

The following example shows that the conditions in Corollary 3.6 hold generically.

Example 3.7. Let $\sigma_q\colon \mathrm{GL}(V) \to \mathrm{GL}(V)$ be a standard Frobenius endomorphism which stabilises a connected reductive subgroup H of $\mathrm{GL}(V)$ and a maximal torus T of H. Pick $l \in \mathbb{N}$ such that firstly all the different T-weights of V are still distinct when restricted to $T_{\sigma_q^l}$ and secondly there is a $t \in T_{\sigma_q^l}$, such that $C_{\mathrm{GL}(V)}(T) = C_{\mathrm{GL}(V)}(t)$. Then for every $n \geq l$, both conditions in Corollary 3.6 are satisfied for $\sigma = \sigma_q^n$. Thus there are only finitely many powers of σ_q for which part (i) can fail. The argument here readily generalises to a Steinberg endomorphism of a connected reductive G which induces a generalised Frobenius morphism on H.

4. Overgroups of type A_1

4.1. **Overgroups of type A_1.** We begin with the distinguished case. There are only a few instances when G is simple, p is bad for G, and G admits a distinguished unipotent element of order p. We recall the relevant results concerning the existence of A_1 overgroups of such elements from [26].

Lemma 4.1 ([26, Lem. 4.1])**.** *Let G be simple classical of type B_l, C_l, or D_l and suppose $p = 2$. Then G admits a distinguished involution u if and only if G is of type C_2 and u belongs to the subregular class $\mathscr{C}$ of G. If σ is id_G or a q-Frobenius endomorphism of G and $u \in \mathscr{C} \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u.*

Example 4.2. Let G be simple of type C_2, $p = 2$, and suppose u is a distinguished unipotent of order 2 in G. Let σ be id_G or a q-Frobenius endomorphism of G. Then Lemma 4.1 provides a σ-stable subgroup A of type A_1 containing u. Any such subgroup A is not G-ir, thanks to [16, Prop. 1.2]. In fact, according to *loc. cit.*, there are two G-conjugacy classes of such A_1 subgroups in G; see also Example 4.17 below. Since A is contained in a proper parabolic subgroup of G, so is A_σ. So the latter is also not G-ir.

Lemma 4.3 ([26, Lem. 3.3, Lem. 4.2])**.** *Let G be simple of exceptional type and suppose p is bad for G. Then G admits a distinguished unipotent element u of order p if and only if G is of type G_2, $p = 3$, and either u belongs to the subregular class $G_2(a_1)$*[1] *or to the class $A_1^{(3)}$ of G. Moreover, if σ is id_G or a q-Frobenius endomorphism of G and $u \in G_2(a_1) \cap G_\sigma$, then there exists a σ-stable subgroup A of G of type A_1 containing u. In case $u \in A_1^{(3)}$, there is no overgroup of u in G of type A_1.*

[1]Throughout, we use the Bala–Carter notation for distinguished classes in the exceptional groups, cf. [9, §5.9].

Example 4.4. Let G be simple of type G_2 and $p = 3$. Let H be a reductive subgroup of G containing a distinguished unipotent element u from G. Then, as $p = 3 = a(G_2)$, it follows from Theorem 3.1 that H° is G-ir, and so is H. This applies in particular to the subgroup A of G of type A_1 containing u when $u \in G_2(a_1)$. Since 3 is not a good prime for G, Theorem 1.1 does not apply in this case. See also [34, Cor. 2].

In case of the presence of a q-Frobenius endomorphism of G stabilizing H, we show in our proof of Theorem 1.5 that H_σ is also G-ir.

The existence of A_1 overgroups for unipotent elements of order p is guaranteed by the following fundamental results of Testerman [36, Thm. 0.1] if p is good for G and else by Proud–Saxl–Testerman [26].

Theorem 4.5 ([36, Thm. 0.1, Thm. 0.2])**.** *Let G be a semisimple group and suppose p is good for G. Let σ be id_G or a Steinberg endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

Testerman's proof of Theorem 4.5 is based on case-by-case checks and depends in part on computer calculations involving explicit unipotent class representatives. For a uniform proof of the theorem, we refer the reader to McNinch [21].

Theorem 4.6 ([26, Thm. 5.1])**.** *Let G be semisimple and suppose p is bad for G. Let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. If $p = 3$, and G has a simple component of type G_2, assume that the projection of u into this component does not lie in the class $A_1^{(3)}$. Then there exists a σ-stable subgroup of G of type A_1 containing u.*

Corollary 4.7. *Let G be simple of type G_2, $p = 3$ and let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in A_1^{(3)} \cap G_\sigma$. Then there is no proper semisimple subgroup H of G containing u. In particular, any such u is* semiregular*, that is, $C_G(u)$ does not contain a non-central semisimple element of G.*

Proof. By way of contradiction, suppose H is a proper semisimple subgroup of G containing u. Since $p = 3$ is good for H (e.g., see [34, Cor. 3]), there is a σ-stable A_1 subgroup A in H containing u, by Theorem 4.5. It follows from Lemma 4.3 that $u \in G_2(a_1)$ which contradicts the hypothesis that $u \in A_1^{(3)}$. ☐

With Theorem 4.6 in hand, we need to deal next with the case when $p = 3$ is good for G and H admits a simple component of type G_2.

Lemma 4.8. *Suppose G is simple and $p = 3$ is good for G. Let H be a connected reductive subgroup of G. Let $u \in H$ be a unipotent element of order 3. Then H does not admit a simple component of type G_2.*

Proof. (cf. [16, p. 387]) Since p is good for G, G is simple of classical type. Let V be the natural module for G. Since u has order 3, the largest Jordan block size of u on V is at most 3. Since u is distinguished in G, the Jordan block sizes of u are distinct and of the same parity. Hence $\dim V \leq 4$. Since a non-trivial representation of a simple algebraic group of type G_2 has dimension at least 5, H does not have a simple component of type G_2. ☐

In summary, we see that if $u \in G$ has order p then u is contained in an A_1 subgroup of G unless $p = 3$ and G has a simple G_2 factor such that the projection of u onto this factor lies in the class $A_1^{(3)}$.

4.2. **Good A_1 overgroups.** In his seminal work [28], Seitz defines an important class of A_1 overgroups of a unipotent element of order p. He establishes the existence and fundamental properties of these overgroups provided p is good for G.

Definition 4.9. Following [28], we call a subgroup A of G of type A_1 a *good* A_1 subgroup of G if the weights of a maximal torus of A on $\mathrm{Lie}(G)$ are at most $2p-2$. Else we call A a *bad* A_1 subgroup of G. This is of course independent of the choice of a maximal torus of A. For u a unipotent element of G of order p, we define

$$\mathscr{A}(u) := \mathscr{A}_G(u) := \{A \subseteq G \mid A \text{ is a good } A_1 \text{ subgroup of } G \text{ containing } u\}$$

and analogously, for a connected reductive subgroup M of G we write $\mathscr{A}_M(u)$ for the set of all good A_1 subgroups of M containing u.

If $A \subseteq H \subseteq G$ are connected reductive groups such that A is a good A_1 in G, then A is obviously also a good A_1 in H.

We record parts of the main theorems from [28] for our purposes, using the notation above.

Theorem 4.10 (cf. [28, Thm. 1.1, Thm. 1.2])**.** *Let G be simple. Suppose p is good for G and u is a unipotent element of G of order p. Then the following hold:*

- (i) $\mathscr{A}(u) \neq \varnothing$.
- (ii) $R_u(C_G(u))$ *acts transitively on* $\mathscr{A}(u)$.
- (iii) *Let* $A \in \mathscr{A}(u)$. *Then* $C_G(u)$ *is the semidirect product of* $C_G(A)$ *and* $R_u(C_G(u))$ *and* $C_G(A)$ *is reductive.*
- (iv) *Let* $A \in \mathscr{A}(u)$. *Then* A *is* G*-completely reducible.*

The following example is a special instance of Theorem 4.10.

Example 4.11. Let V be an $\mathrm{SL}_2(k)$-module such that weights of a maximal torus T of $\mathrm{SL}_2(k)$ on V are strictly less than p. Then the weights of T in the induced action on $\mathrm{Lie}(\mathrm{GL}(V)) \cong V \otimes V^*$ are at most $2p-2$. Thus the induced subgroup A in $\mathrm{GL}(V)$ is a good A_1. In this situation the highest weights of T on each composition factor of V are restricted, so V is a semisimple $\mathrm{SL}_2(k)$-module; cf. [1, Cor. 3.9].

In the next theorem we recall parts of the analogue of Theorem 4.10 for finite overgroups of type A_1.

Theorem 4.12 (cf. [28, Thm. 1.4])**.** *Let G be simple. Suppose p is good for G. Let $\sigma : G \to G$ be a Steinberg endomorphism of G. Suppose $u \in G_\sigma$ is unipotent of order p.*

- (i) $\mathscr{A}(u)_\sigma \neq \varnothing$.
- (ii) $R_u(C_G(u))_\sigma$ *acts transitively on* $\mathscr{A}(u)_\sigma$.
- (iii) *Let* $A \in \mathscr{A}(u)_\sigma$. *Suppose that* $q > 7$ *if* G *is of exceptional type. Then* A_σ *is* σ*-completely reducible.*

Remark 4.13. Parts (i) and (ii) of Theorem 4.12 follow from parts (i) and (ii) of Theorem 4.10 and the Lang–Steinberg Theorem, see [28, Prop. 9.1].

Remark 4.14. (i). Concerning the terminology in Theorem 4.12(iii), following [13], a subgroup H of G is said to be σ*-completely reducible*, provided that whenever H lies in a σ-stable parabolic subgroup P of G, it lies in a σ-stable Levi subgroup of P. This notion is motivated by certain rationality questions concerning G-complete reducibility; see [13] for details. For a σ-stable subgroup H of G, this property is equivalent to H being G-cr, thanks to [13, Thm. 1.4]. Thus in turn, by the latter, Theorem 4.12(iii) follows from Theorem 4.10(iv).

(ii). Apart from the special conjugacy class of good A_1 subgroups in G asserted in Theorem 4.10, there might be a plethora of conjugacy classes of bad A_1 subgroups in G even when p is good for G. Just take a non-semisimple representation $\rho : \mathrm{SL}_2(k) \to \mathrm{SL}(V) = G$ in characteristic $p > 0$. Then the A_1 subgroup $\rho(\mathrm{SL}_2(k))$ is bad in G, while p is good for G. For a concrete example, see [15, Rem. 5.12]. This can only happen if p is sufficiently small compared to the rank of G, thanks to Theorem 3.3.

(iii). The proofs of Theorems 4.10 and 4.12 by Seitz in [28] depend on separate considerations for each Dynkin type and involve in part intricate arguments for the component groups of centralizers of unipotent elements. In [22], McNinch presents uniform proofs of Seitz's theorems which are almost entirely free of any case-by-case checks, utilizing methods from geometric invariant theory. However, McNinch's argument (see [22, Thm. 44]) of the conjugacy result in Theorem 4.10(ii) depends on the fact that for a good A_1 subgroup A of G, the A-module $\mathrm{Lie}(G)$ is a tilting module. The latter is established by Seitz in [28, Thm. 1.1].

We present some alternative short and uniform proofs of parts (i)–(iii) of Theorem 4.10 based on the results on associated cocharacters from §2.8. For part (iii) we require the following fact.

Lemma 4.15. *Suppose the connected reductive group H acts on the affine variety X. Suppose some Borel subgroup B of H fixes a point $x \in X$. Then H fixes x.*

Proof. Because B fixes x, there is an induced map $H/B \to X$ given by $hB \mapsto h \cdot x$. Since H/B is projective and irreducible, this map is constant. Hence H fixes x. □

Proof of Theorem 4.10(i)–(iii). According to the Bala–Carter Theorem 2.1, there is a Levi subgroup L of G such that u is distinguished in L (including the case $L = G$). Since p is also good for L (cf. §2.2), there is an A_1 subgroup A in $\mathscr{D}L$ containing u, owing to Theorem 4.5. Since p is good for A, there is a cocharacter λ in $\Omega^a_A(u)$, by Proposition 2.6(i). In particular, $\lambda(\mathbb{G}_m)$ is a maximal torus in A. Then λ belongs to $\Omega^a_L(u)$, by Lemma 2.11 and thanks to Lemma 2.12, it also belongs to $\Omega^a_G(u)$. It thus follows from Lemma 2.10 that the weights of $\lambda(\mathbb{G}_m)$ on $\mathrm{Lie}(G)$ do not exceed $2p-2$. Thus A is a good A_1 subgroup of G, and so Theorem 4.10(i) follows.

Next let $A, \tilde{A}$ be in $\mathscr{A}(u)$. Let $\lambda \in \Omega^a_A(u)$ and $\tilde{\lambda} \in \Omega^a_{\tilde{A}}(u)$. By the arguments above, we have $\Omega^a_A(u) \cup \Omega^a_{\tilde{A}}(u) \subseteq \Omega^a_G(u)$. Thanks to Proposition 2.6(ii), λ and $\tilde{\lambda}$ are $C_G(u)^\circ$-conjugate. Further, by Proposition 2.8, we have $C_G(u)^\circ = C_G(\lambda, u)^\circ R_u(C_G(u))$, so that λ and $\tilde{\lambda}$ are $R_u(C_G(u))$-conjugate. There is no harm in assuming that $\lambda = \tilde{\lambda}$. But then A and $\tilde{A}$ share the common Borel subgroup $B = \lambda(\mathbb{G}_m)U$, where U is the unique 1-dimensional subgroup in A (and $\tilde{A}$) containing u and normalized by $\lambda(\mathbb{G}_m)$. It follows from [19, Lem. 2.4] that $A = \tilde{A}$. So Theorem 4.10(ii) holds.

Now let A be in $\mathscr{A}(u)$ and let $\lambda \in \Omega^a_A(u)$. Then by the argument above, $\lambda \in \Omega^a_G(u)$. Clearly, $C_G(A) \subseteq C_G(\lambda, u)$. Let $g \in C_G(\lambda, u)$. Then gAg^{-1} and A share the common Borel subgroup $B = \lambda(\mathbb{G}_m)U$, as above. Once again, thanks to [19, Lem. 2.4], we conclude that $gAg^{-1} = A$, and so $g \in N_G(A)$. Let A act on G by conjugation. Since $C_G(B) = C_G(\lambda, u)$

(cf. (2.7)), B is contained in $C_A(g)$. It follows from Lemma 4.15 that $g \in C_G(A)$. Consequently, $C_G(\lambda, u) \subseteq C_G(A)$. Thus we have $C_G(\lambda, u) = C_G(A)$. Finally, using Proposition 2.8, we get that $C_G(u) = C_G(A)R_u(C_G(u))$ is a semidirect product and $C_G(A)$ is reductive. Theorem 4.10(iii) follows. □

For a short and uniform proof of Theorem 4.10(iv), see [22, Thm. 52].

The following relates the set of cocharacters of G which are associated to a unipotent element u of G of order p to those stemming from good A_1 overgroups of u in G.

Corollary 4.16. *Let G be simple. Suppose p is good for G and $u \in G$ is unipotent of order p.*

(i) *Suppose u is distinguished in the Levi subgroup L of G (cf. Theorem 2.1). Then* **any** *A_1 subgroup of L containing u belongs to $\mathscr{A}(u)$. In particular,* **any** *A_1 subgroup of L containing u belongs to $\mathscr{A}_L(u)$.*

(ii) *$\mathscr{A}(u)$ coincides with the set of all $C_G(u)^\circ$-conjugates of an A_1 subgroup from part (i).*

(iii) *We have a disjoint union*

$$\Omega_G^a(u) = \dot{\bigcup_{A \in \mathscr{A}(u)}} \Omega_A^a(u).$$

(iv) *Let $A \in \mathscr{A}(u)$ and let $\lambda \in \Omega_A^a(u)$. Then $C_G(A) = C_G(\lambda, u)$.*

(v) *Let $\lambda \in \Omega_G^a(u)$. Then $C_G(\lambda, u)$ is G-completely reducible.*

(vi) *Let $A \in \mathscr{A}(u)$ and let $\lambda \in \Omega_A^a(u)$. Then we have*

$$C_G(u)/C_G(u)^\circ \cong C_G(\lambda, u)/C_G(\lambda, u)^\circ = C_G(A)/C_G(A)^\circ.$$

Proof. Part (i) is immediate from the previous proof.

(ii). Let u, L, and $A \in \mathscr{A}_L(u)$ be as in part (i). Then for any $g \in C_G(u)^\circ$, the conjugate gAg^{-1} of A is a good A_1 subgroup of gLg^{-1} and thus of G, as u is also distinguished in gLg^{-1}. Thus any $C_G(u)^\circ$-conjugate of A is again a good A_1 subgroup of G. Now let $\tilde{A}$ be in $\mathscr{A}(u)$. Theorem 4.10(ii) asserts that there is a $g \in C_G(u)^\circ$ such that $\tilde{A} = gAg^{-1}$.

(iii). We first prove that the union in (iii) is disjoint. The argument is more less identical to the proof of Theorem 4.10(ii) above. Let $A, \tilde{A} \in \mathscr{A}(u)$ and let $\lambda \in \Omega_A^a(u) \cap \Omega_{\tilde{A}}^a(u)$. Then A and $\tilde{A}$ share the common Borel subgroup $\lambda(\mathbb{G}_m)U$, where U is the unique 1-dimensional subgroup in A and $\tilde{A}$ containing u. It follows from [19, Lem. 2.4] that $A = \tilde{A}$.

Thanks to parts (i) and (ii) and the proof of Theorem 4.10(i) above, we get $\dot{\bigcup}_{A \in \mathscr{A}(u)} \Omega_A^a(u) \subseteq \Omega_G^a(u)$. Conversely, let $\lambda \in \Omega_G^a(u)$. By Proposition 2.6(ii), λ is $C_G(u)^\circ$-conjugate to a member of this union.

(iv). This equality is derived in our proof of Theorem 4.10(iii) above.

(v). By part (iii), there is an $A \in \mathscr{A}(u)$ such that $\lambda \in \Omega_A^a(u)$. Owing to Theorem 4.10(iv), A is G-cr, thus so is $C_G(A)$, by [6, Cor. 3.17]. The result now follows from part (iv).

(vi). This follows from Proposition 2.8 and (iv). □

We note that Corollary 4.16(i) is implicit in [28] and Corollary 4.16(iii) is stated in [22, p. 393].

In [28, §9], Seitz exhibits instances when there is no good A_1 overgroup of an element of order p when p is bad for G. As we explain next, Example 4.2 gives a counterexample to Theorem 4.10(iv) in case p is bad for G: that is, it gives a good A_1 subgroup A containing a unipotent element of order p of G such that A is not G-cr. Specifically, we show that some of the A_1 subgroups in that example are good A_1 subgroups of G, but thanks to Example 4.2, they are not G-cr.

Example 4.17 (Example 4.2 continued)**.** Let G be simple of type C_2 and $p = 2$. Let σ be id_G or a q-Frobenius endomorphism of G. Let $\mathscr{C}$ denote the subregular unipotent class of G. Suppose $u \in \mathscr{C} \cap G_\sigma$. Then the σ-stable subgroup A of G of type A_1 containing u, guaranteed by Lemma 4.1, is not G-cr, by Example 4.2. Let E be the natural module for $\mathrm{SL}_2(k)$. Then there are two conjugacy classes of embeddings of $\mathrm{SL}_2(k)$ into $G = \mathrm{Sp}(V)$: we can take either $V \cong E \oplus E$ or $V \cong E \otimes E$, as an $\mathrm{SL}_2(k)$-module. The images of both embeddings meet the class $\mathscr{C}$ non-trivially. One checks that the highest weight of a maximal torus of $\mathrm{SL}_2(k)$ on $\mathrm{Lie}(G)$ is 4 in the second instance. So in this case the image of $\mathrm{SL}_2(k)$ in G is not a good A_1. In contrast, in the first instance the highest weight of a maximal torus of $\mathrm{SL}_2(k)$ on $\mathrm{Lie}(G)$ is $2 = 2p - 2$, by Example 4.11. So the image of $\mathrm{SL}_2(k)$ in G is a good A_1 in $\mathrm{SL}(V)$, and so it is a good A_1 in G as well.

5. Proofs of Theorems 1.1 and 1.3–1.5

Armed with the results on associated cocharacters from above, we prove Theorems 1.1 and 1.4 simultaneously.

Proof of Theorems 1.1 and 1.4. First we reduce to the case when G is simple. In view of Lemma 3.2, it suffices to show that H (resp. H_σ) is G-cr. Thus we need to show that G-complete reducibility and the property of being distinguished behave well with respect to the steps in this reduction process. This is achieved by [6, Lem. 2.12] and [15, §4.3], respectively. For, let $\pi\colon G \to G/Z(G)^\circ$ be the canonical projection. Owing to [6, Lem. 2.12(ii)(b)] and [15, §4.3], we can replace G with $G/Z(G)^\circ$, so without loss we can assume that G is semisimple. Let $G_1, \dots, G_r$ be the simple factors of G. Multiplication gives an isogeny from $G_1 \times \cdots \times G_r$ to G. Thus, again by [6, Lem. 2.12(ii)(b)] and [15, §4.3], we can replace G with $G_1 \times \cdots \times G_r$, so we can assume G is the product of its simple factors. Finally, thanks to [6, Lem. 2.12(i)] and [15, §4.3], it is thus enough to prove the result when G is simple. We may also assume that H is connected and semisimple, since any unipotent element of H° is contained in the derived subgroup $\mathscr{D}H^\circ$, and H is G-ir if $\mathscr{D}H^\circ$ is.

Since u has order p also in H, there is an overgroup A of u of type A_1 inside H (resp. σ-stable A_1 subgroup of H), thanks to Theorem 4.5 in case p is good for H, and else by Theorem 4.6 and Lemma 4.8. Since p is good for A, there is a cocharacter λ in $\Omega_A^a(u)$, by Proposition 2.6(i). Since u is distinguished in G, λ belongs to $\Omega_G^a(u)$, by Lemma 2.11. It follows from Lemma 2.10 that A is a good A_1 subgroup of G (resp. a good σ-stable A_1 subgroup of G). Thus, by Theorem 4.10(iv), A is G-cr (resp. by Theorem 4.12(iii), A_σ is σ-completely reducible and so, thanks to Remark 4.14(i), A_σ is G-cr). Finally, by Lemma 3.2, A (resp. A_σ) is G-ir and so is H (resp. H_σ). □

As a consequence of Theorems 1.1 and 1.4 we obtain the following.

Corollary 5.1. *Let G be a connected reductive group. Suppose p is good for G. Let σ be id_G or a q-Frobenius endomorphism of G. Let $u \in G_\sigma$ be unipotent of order p. Suppose u is distinguished in the σ-stable Levi subgroup L of G (cf. Remark 2.2(ii)). Let H be a σ-stable connected reductive subgroup of L containing u, then H_σ is G-completely reducible.*

Proof. As p is also good for L (cf. §2.2), it follows from Theorem 1.1 (resp. 1.4) applied to L that H_σ is L-ir and so is L-cr. Thus, H_σ is G-cr, by [30, Prop. 3.2]. □

Finally, we address Theorems 1.3 and 1.5.

Proof of Theorems 1.3 and 1.5. By Theorems 1.1 and 1.4, the only cases we need to consider are when p is bad for G. If G is classical, then we are in the situation of Lemma 4.1 and Example 4.2.

We are left to consider the case when G is of exceptional type. Then owing to Lemma 4.3, G is of type G_2 and $p = 3$. There is no harm in assuming that H is semisimple. It follows from Example 4.4 that H is G-ir. Thus Theorem 1.3 follows. So consider the setting of Theorem 1.5 when $\sigma|_H$ is a q-Frobenius endomorphism of H in this case. By Corollary 4.7, u belongs to the subregular class of G_2. It follows from the proof of Lemma 4.3 in [26] that u is contained in a σ-stable maximal rank subgroup of G of type $A_1\tilde{A}_1$ and this type is unique. Since H is proper and semisimple, $H \subseteq M$, where M is a σ-stable maximal rank subgroup of G of type $A_1\tilde{A}_1$. Since p is good for H, there is a σ-stable subgroup A of H of type A_1 containing u, by Theorem 4.5. Thus $A \subseteq H \subseteq M$. Since u is also distinguished in M and $p = 3$ is good for M, Theorem 1.4 shows that A_σ is M-ir. Note that M is the centralizer of a semisimple element of G of order 2 (by Deriziotis' Criterion, cf. [10, 2.3]). Since A_σ is M-cr, it is G-cr, owing to [6, Cor. 3.21]. Once again, by Lemma 3.2, A_σ is G-ir and so is H_σ. Theorem 1.5 follows. □

Remark 5.2. In [16, §7], Korhonen gives counterexamples to Theorem 1.1 when the order of the distinguished unipotent element of G is greater than p (even when p is good for G [16, Prop. 7.1]). Theorem 3.1 implies that this can only happen when $p < a(G)$. For instances of overgroups of distinguished unipotent elements of G of order greater than p for $p \geq a(G)$ (and p good for G), so that Theorem 3.1 applies, see Examples 5.4 and 5.5.

Remark 5.3. In view of Remark 5.2, it is natural to ask for instances of G, u and H when the conclusion of Theorem 3.1 holds even when $p < a(G)$ but p is still good for G. If p is good for G and G is simple classical, non-regular distinguished unipotent elements always belong to a maximal rank semisimple subgroup H of G, by [36, Prop. 3.1, Prop. 3.2]. For G simple of exceptional type this is also the case in almost all instances of non-regular distinguished unipotent elements, cf. [36, Lem. 2.1]. Each such H is obviously G-irreducible. This is independent of p of course and thus applies in particular when $p < a(G)$. For instance, let G be of type E_7, $p = 5$, and suppose u belongs to the distinguished class $E_7(a_3)$ (resp. $E_7(a_4)$, $E_7(a_5)$). Then $\mathrm{ht}_J(\rho) = 9$ (resp. 7, 5), so u has order 5^2, by Lemma 2.3 in each case. Since u does not have order 5, Theorem 1.1 does not apply, and since $5 < 8 = a(G)$ neither does Theorem 3.1. Nevertheless, in each case u is contained in a maximal rank subgroup H of type A_1D_6, cf. [36, p. 52], and each such H is G-ir.

We close with several additional higher order examples in good characteristic when Theorem 1.1 does not apply but Theorem 3.1 does.

Example 5.4. Let G be of type E_6. Suppose p is good for G. In [36, Lem. 2.7], Testerman exhibits the existence of a simple subgroup H of G of type C_4 whose regular unipotent class belongs to the subregular class $E_6(a_1)$ of G. Let u be regular unipotent in H. For $p = 7$, the order of u is 7^2, by Lemma 2.3, so Theorem 1.1 can't be invoked to say anything about H. However, for $p = 7 = a(G)$, we infer from Theorem 3.1 that H is G-ir.

Example 5.5. Let G be of type E_8. Suppose $p = 11$. Let u be in the distinguished class $E_8(a_3)$ (resp. $E_8(a_4)$, $E_8(b_4)$, $E_8(a_5)$, or $E_8(b_5)$). From the corresponding weighted Dynkin diagram corresponding to u we get $\mathrm{ht}_J(\rho) = 17$ (resp. 14, 13, 11, or 11), cf. [9, p. 177]. It follows from Lemma 2.3 that in each of these instances u has order 11^2. So we can't appeal to Theorem 1.1 to deduce anything about reductive overgroups of u. But as $11 = p \geq a(G) = 9$, Theorem 3.1 applies and allows us to conclude that each such overgroup is G-ir. For example, in each instance, u is contained in a maximal rank subgroup H of G of type A_1E_7 or D_8, cf. [36, p. 52].

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