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From optimal martingales to randomized dual optimal stopping

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Abstract

In this article we study and classify optimal martingales in the dual formulation of optimal stopping problems. In this respect we distinguish between weakly optimal and surely optimal martingales. It is shown that the family of weakly optimal and surely optimal martingales may be quite large. On the other hand it is shown that the Doob-martingale, that is, the martingale part of the Snell envelope, is in a certain sense the most robust surely optimal martingale under random perturbations. This new insight leads to a novel randomized dual martingale minimization algorithm that doesn't require nested simulation. As a main feature, in a possibly large family of optimal martingales the algorithm efficiently selects a martingale that is as close as possible to the Doob martingale. As a result, one obtains the dual upper bound for the optimal stopping problem with low variance.

1 Introduction

The last decades have seen a huge development of numerical methods for solving optimal stopping problems. Such problems became very prominent in the financial industry in the form of American derivatives. For such derivatives one needs to evaluate the right of exercising (stopping) a certain cash-flow (reward) process Z at some (stopping) time τ, up to some time horizon T. From a mathematical point of view this evaluation comes down to solving an optimal stopping problem

$$Y^\star = \sup_{\text{stopping time } \tau \leq T} \mathsf{E}[\underbrace{Z_\tau}_{\text{reward at stopping}}].$$

Typically the cash-flow Z depends on various underlying assets and/or interest rates and as such is part of a high dimensional Markovian framework. Particularly for high dimensional stopping problems, virtually all generic numerical solutions are Monte Carlo based. Most of the first numerical solution approaches were of primal nature in the sense that the goal was to construct a "good" exercise policy and to simulate a lower biased estimate of $Y^\star$. In this respect we mention, for example, the well-known regression methods by Longstaff & Schwartz [11], Tsiklis & Van Roy [14], and the stochastic mesh approach by Broadie & Glasserman [5], and the stochastic policy improvement method by Kolodko & Schoenmakers [10]. For further references we refer to the literature, for example [8] and the references therein.

In this paper we focus on the dual approach developed by Rogers [12], and Haugh & Kogan [9], initiated earlier by Davis & Karatzas [6]. In the dual method the stopping problem is solved by minimizing over a set of martingales, rather than a set of stopping times,

$$Y^\star = \inf_{M:\text{ martingale, } M_0=0} \mathsf{E}\left[\max_{0\leq s\leq T}(Z_s - M_s)\right]. \tag{1.1}$$

A canonical minimizer of this dual problem is the martingale part, $M^\star$ of the Doob(-Meyer) decomposition of the Snell envelope

$$Y_t^\star = \sup_{t\leq \text{stopping time } \tau\leq T} \mathsf{E}_{\mathcal{F}_t}\left[Z_\tau\right],$$

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which moreover has the nice property that

$$Y_0^\star = \max_{0\leq s\leq T}(Z_s - M_s^\star) \quad \text{almost surely.} \tag{1.2}$$

That is, if one would succeed in finding $M^\star$, the value of $Y^\star$ can be obtained from one trajectory of $Z - M^\star$ only.

Shortly after the development of the duality method in [12] and [9], various numerical approaches for computing dual upper bounds for American options based on it appeared. May be one of the most popular methods is the nested simulation approach by Andersen & Broadie [1], who essentially construct an approximation to the Doob martingale of the Snell envelope via stopping times obtained by the Longstaff & Schwartz method [11]. A few years later, a linear Monte Carlo method for dual upper bounds was proposed in [3]. In fact, as a common feature, both [1] and [3] aimed at constructing (an approximation of) the Doob martingale of the Snell envelope via some approximative knowledge of continuation functions obtained by the method of Longstaff & Schwartz or in another way. Instead of relying on such information, the common goal in later studies [7], [13], [2], [4], was to minimize the expectation functional in the dual representation (1.1) over a linear space of generic "elementary" martingales. Indeed, by parameterizing the martingale family in a linear way and replacing the expectation in (1.1) by the sample mean over a large set of trajectories, the resulting minimization comes down to solving a linear program. However, it was pointed out in [13] that in general there may exist martingales that are "weakly" optimal in the sense that they minimize (1.1), but fail to have the "almost sure property" (1.2). As a consequence, the estimator for the dual upper bound due to such martingales may have high variance. Moreover, an example in [13] illustrates that a straightforward minimization of the sample mean corresponding to (1.1) may end up with a martingale that is asymptotically optimal in the sense of (1.1) but not surely optimal in the sense of (1.2), when the sample size tends to infinity. As a remedy to this problem, in [2] variance penalization is proposed, whereas in [4] the sample mean is replaced by the maximum over all trajectories.

In this paper we first extend the study of surely optimal martingales in [13] to the larger class of *weakly* optimal martingales. As a principal contribution, we give a complete characterization of weakly and surely optimal martingales and moreover consider the notion of randomized dual martingales. In particular, it is shown that in general there may be a fullness of martingales that are optimal but not surely optimal. In fact, straightforward minimization procedures based on the sample mean in (1.1) may typically return martingales of this kind, even if the Doob martingale of the Snell envelope is contained in the martingale family (as illustrated already in [13], though at a somewhat pathological example with partially deterministic cash-flows). As another main contribution we will show that the Doob martingale plays a distinguished role within the family of all optimal martingales. Namely, it will be shown that by randomizing the arguments in the path-wise maximum for each trajectory in a particular way, any non-Doob optimal martingale can be turned to a suboptimal one. More specifically, we will prove that there exists a particular "optimal randomization" such that the Doob martingale, perturbed or randomized with it, remains guaranteed (surely) optimal, while any other surely or weakly optimal martingale turns to a suboptimal one. Of course, as a rule this "optimal randomization" is not directly known or available in practical applications. But, it turns out that by just incorporating some simple randomization due to uniform random variables, sample mean minimization may return a martingale that is closer to the Doob-martingale than one obtained without randomization. We thus end up with a martingale with low variance, which in turn guarantees that the corresponding upper bound based on (1.1) is tight (see [2] and[13]). Compared to [4] and [2], the benefit of this new randomized dual approach is its computational efficiency: From the experiments we conclude that it may be sufficient to add on for each trajectory simple i.i.d. uniform random variables to (some of) the arguments of the maximum. An extensive numerical analysis of the here presented randomized dual martingale approach will certainly be an interesting subsequent study but is considered beyond the scope of this article.

The structure of the paper is as follows. Section 2 carries out a systematic theoretical analysis of optimal martingales. In Section 3 we deal with randomized optimal martingales and the effect of randomizing the Doob-martingale. More technical proofs are given in Section 4 and some first numerical examples are presented in Section 5.

2 Characterization of optimal martingales

Since practically any numerical approach to optimal stopping is based on a discrete exercise grid, we will work within in a discrete time setup. That is, it is assumed that exercise (or stopping) is restricted to a discrete set of exercise times $t_0 = 0, ..., t_J = T$, for some time horizon T and some $J \in \mathbb{N}_+$. For notational convenience we will further identify the exercise times t_j with their index j, and thus monitor the reward process Z_j, at the "times" $j = 0, ..., J$.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a filtered probability space with discrete filtration $\mathcal{F} = (\mathcal{F}_j)_{j\geq 0}$. An optimal stopping problem is a problem of stopping the reward process $(Z_j)_{j\geq 0}$ in such a way that the expected reward is maximized. The value of the optimal stopping problem with horizon J at time $j \in \{0, \ldots, J\}$ is given by

$$Y_j^\star = ess\ sup_{\tau\in\mathcal{T}[j,\ldots,J]} \mathsf{E}_{\mathcal{F}_j}[Z_\tau], \tag{2.1}$$

provided that Z was not stopped before j. In (2.1), $\mathcal{T}[j, \ldots, J]$ is the set of $\mathcal{F}$-stopping times taking values in $\{j, \ldots, J\}$ and the process $\left(Y_j^\star\right)_{j\geq 0}$ is called the Snell envelope. It is well known that $Y^\star$ is a supermartingale satisfying the backward dynamic programming equation (Bellman principle):

$$Y_j^\star = \max\left(Z_j, \mathsf{E}_{\mathcal{F}_j}[Y_{j+1}^\star]\right), \quad 0 \leq j < J, \quad Y_J^\star = Z_J.$$

Along with a primal approach based on the representation (2.1), a dual method was proposed in [12] and [9]. Below we give a short self contained recap while including the notions of *weak* and *sure* optimality.

Let $\mathcal{M}$ be the set of martingales M adapted to $\mathcal{F}$ with $M_0 = 0$. By using the Doob's optimal sampling theorem one observes that

$$Y_j^\star \leq \mathsf{E}_{\mathcal{F}_j}\left[\max_{j\leq r\leq J}(Z_r - M_r + M_j)\right], \quad j = 0, \ldots, J, \tag{2.2}$$

for any $M \in \mathcal{M}$. We will say that a martingale M is *weakly optimal*, or just *optimal*, at j, for some $j = 0, ..., J$, if

$$Y_j^\star = \mathsf{E}_{\mathcal{F}_j}\left[\max_{j\leq r\leq J}(Z_r - M_r + M_j)\right]. \tag{2.3}$$

The set of all martingales (weakly) optimal at j will be denoted by $\mathcal{M}^{\circ,j}$. The set of martingales optimal at j for all $j = 0, \ldots, J$, is denoted by $\mathcal{M}^\circ$. We say that a martingale M is *surely optimal* at j, for some $j = 0, ..., J$, if

$$Y_j^\star = \max_{j\leq r\leq J}(Z_r - M_r + M_j) \quad \text{almost surely.} \tag{2.4}$$

The set of all surely optimal martingales at j will be denoted by $\mathcal{M}^{\circ\circ,j}$. The set of surely optimal martingales at j for all $j = 0, \ldots, J$, is denoted by $\mathcal{M}^{\circ\circ}$. Note that, obviously, $\mathcal{M}^{\circ\circ} \subset \mathcal{M}^\circ \subset \mathcal{M}$.

Now there always exists at least one surely optimal martingale, the so-called Doob-martingale coming from the Doob decomposition of the Snell envelope $(Y_j^\star)_{j\geq 0}$. Indeed, consider the Doob decomposition of $Y^\star$, that is,

$$Y_j^\star = Y_0^\star + M_j^\star - A_j^\star, \tag{2.5}$$

where $M^\star$ is a martingale with $M_0^\star = 0$, and $A^\star$ is predictable with $A_0^\star = 0$. It follows immediately that

$$M_j^\star = \sum_{l=1}^{j} (Y_l^\star - \mathsf{E}_{\mathcal{F}_{l-1}}[Y_l^\star]), \quad A_j^\star = \sum_{l=1}^{j} (Y_{l-1}^\star - \mathsf{E}_{\mathcal{F}_{l-1}}[Y_l^\star]), \tag{2.6}$$

and so $A^\star$ is non-decreasing due to the fact that $Y^\star$ is a supermartingale. One thus has by (2.5) on the one hand

$$\max_{j\leq r\leq J}(Z_r - M_r^\star + M_j^\star) = Y_j^\star + \max_{j\leq r\leq J}(Z_r - Y_r^\star + A_j^\star - A_r^\star) \leq Y_j^\star$$

and due to (2.2) on the other hand

$$\mathsf{E}_{\mathcal{F}_j}\left[\max_{j\leq r\leq J}(Z_r - M_r^\star + M_j^\star)\right] \geq Y_j^\star.$$

Thus, it follows that (2.4) holds for arbitrary j, hence $M^\star \in \mathcal{M}^{\circ\circ}$. Furthermore we have the following properties of the sets $(\mathcal{M}^{\circ,j})$ and $(\mathcal{M}^{\circ\circ,j})$.

Proposition 2.1. *The sets $\mathcal{M}^{\circ,j}$ and $\mathcal{M}^{\circ\circ,j}$ for $j = 0, ..., J$, $\mathcal{M}^\circ$, and $\mathcal{M}^{\circ\circ}$ are convex.*

As an immediate consequence of Proposition 2.1; if there exist more than one weakly (respectively surely) optimal martingale, then there exist infinitely many weakly (respectively surely) optimal martingales.

Proposition 2.2. *It holds that $M \in \mathcal{M}^{\circ,j}$ for some $0 \leq j \leq J$, if and only if for any optimal stopping time $\tau_j^\star \geq j$ satisfying*

$$Y_j^\star = \sup_{\tau\geq j} \mathsf{E}_{\mathcal{F}_j}[Z_\tau] = \mathsf{E}_{\mathcal{F}_j}[Z_{\tau_j^\star}],$$

one has that

$$\max_{j\leq r\leq J}(Z_r - M_r) = Z_{\tau_j^\star} - M_{\tau_j^\star}.$$

Proof. Let $\tau_j^\star \geq j$ be an optimal stopping time. Suppose that $M \in \mathcal{M}^{\circ,j}$. On the one hand, one trivially has

$$\max_{j\leq r\leq J}(Z_r - M_r) - \left(Z_{\tau_j^\star} - M_{\tau_j^\star}\right) \geq 0$$

and on the other, since $M \in \mathcal{M}^{\circ,j}$ (see (2.3)),

$$\mathsf{E}_{\mathcal{F}_j}\left[\max_{j\leq r\leq J}(Z_r - M_r) - \left(Z_{\tau_j^\star} - M_{\tau_j^\star}\right)\right] = Y_j^\star - M_j - \left(Y_j^\star - M_j\right) = 0, \quad \text{hence}$$

$$\max_{j\leq r\leq J}(Z_r - M_r) = Z_{\tau_j^\star} - M_{\tau_j^\star} \text{ almost surely.} \tag{2.7}$$

The converse follows from (2.7) by taking conditional $\mathcal{F}_j$-expectations. □

It will be shown below that the class of the optimal martingales $\mathcal{M}^\circ$ may be considerably large. In fact, any such martingale can be seen as a perturbation of the Doob martingale $(M_j^\star)$. For this, let us introduce some further notation and define $\tau^0 := 0^-$ with $0^- < 0$ by convention and let, for $l \geq 1, \tau^l$ be the first optimal stopping time strictly after τ^{l-1}. That is, if $\tau^{l-1} < J$, we define recursively

$$\tau^l = \inf\left\{\tau^{l-1} < i \leq J : Z_i \geq \mathsf{E}_{\mathcal{F}_i}\left[Y_{i+1}^\star\right]\right\},$$

where $Y_{J+1}^\star := 0$. There so will be a last number, l_J say, with $\tau^{l_J} = J$. Further, the family $(\tau_i^\star)_{i\geq 0}$ defined by

$$\tau_i^\star = \tau^l \text{ for } \tau^{l-1} < i \leq \tau^l, \quad l \geq 1, \tag{2.8}$$

is a *consistent optimal stopping family* in the sense that $Y_j^\star = \mathsf{E}_{\mathcal{F}_j}[Z_{\tau_j^\star}]$ and that $\tau_i^\star > i$ implies $\tau_i^\star = \tau_{i+1}^\star$.

The next lemma provides a corner stone for an explicit structural characterization of (weakly) optimal martingales.

Lemma 2.3. *$M \in \mathcal{M}^\circ$ if and only if M is an adapted martingale with $M_0 = 0$ such that the identities*

$$\text{(i)} \quad \max_{\tau^{l-1} < r \leq \tau^l} (Z_r - M_r) = Z_{\tau^l} - M_{\tau^l} \quad \text{if } l \geq 1,$$
$$\text{(ii)} \quad \max_{\tau^{l-1} \leq r \leq \tau^l} (Z_r - M_r) = Z_{\tau^{l-1}} - M_{\tau^{l-1}} \quad \text{if } l > 1$$

hold.

The following lemma anticipates sufficient conditions for a martingale M to be optimal, that is, to be a member of $\mathcal{M}^\circ$.

Lemma 2.4. *Let $(\mathcal{S}_i)_{0\leq i\leq J}$ be an adapted sequence with $\mathcal{S}_0 = 0$ and consider the "shifted" Doob martingale*

$$M_i = M_i^\star - \mathcal{S}_i, \quad 0 \leq i \leq J.$$

Let $l_i \geq 1$ be the unique number such that $\tau^{l_i - 1} < i \leq \tau^{l_i}$ for any $0 \leq i \leq J$. If $\mathcal{S}$ satisfies for all $0 \leq i \leq J$,

$$\max_{\tau^{l_i-1} < r \leq i} (Z_r - Y_r^\star + \mathcal{S}_r - \mathcal{S}_i) \leq 0 \tag{2.9}$$

$$Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y_{\tau^{l_i-1}+1}^\star\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_i \geq 0, \tag{2.10}$$

for $\tau^{l_i-1} < i \leq \tau^{l_i}$ and $l_i > 1$, then M satisfies the identities (i)-(ii) in Lemma 2.3.

Corollary 2.5. *Let us represent an (arbitrary) adapted $\mathcal{S}$ with $\mathcal{S}_0 = 0$ by*

$$\mathcal{S}_{i+1} = \mathcal{S}_i + \zeta_{i+1}, \quad 0 \leq i < J, \tag{2.11}$$

where each ζ_{i+1} is a $\mathcal{F}_{i+1}$-measurable random variable. Then the conditions (2.9) and (2.10) are equivalent to the following ones.

(i) *On the $\mathcal{F}_i$-measurable event $\left\{\tau^{l_i-1} < i < \tau^{l_i}\right\}$ it holds that*

$$\zeta_{i+1} \geq \max_{\tau^{l_i-1} < r \leq i} (Z_r - Y_r^\star + \mathcal{S}_r - \mathcal{S}_i) \quad \text{and} \tag{2.12}$$

$$\zeta_{i+1} \leq Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y_{\tau^{l_i-1}+1}^\star\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_i \quad \text{for } l_i > 1; \tag{2.13}$$

(ii) *On* $\left\{\tau^{l_i} = i\right\}$ *one has that*

$$\zeta_{i+1} \leq Z_i - \mathsf{E}_{\mathcal{F}_i}\left[Y^{\star}_{i+1}\right]. \tag{2.14}$$

Proof. Indeed, take j such that $\left\{\tau^{l_j-1} < j \leq \tau^{l_j}\right\}$, $l_j \geq 1.$ If $j-1 > \tau^{l_j-1}$ then $l_{j-1} = l_j$ and (2.12) and (2.13) imply with $i = j-1$ via (2.11),

$$0 \geq \max_{\tau^{l_j-1}<r\leq j-1} \left(Z_r - Y^{\star}_r + \mathcal{S}_r - \mathcal{S}_j\right) \quad \text{and}$$

$$0 \leq Z_{\tau^{l_j-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_j-1}}}\left[Y^{\star}_{\tau^{l_j-1}+1}\right] + \mathcal{S}_{\tau^{l_j-1}} - \mathcal{S}_j \ \text{ for } \ l_j > 1,$$

respectively, which in turn imply (2.9) (note that $Z_j - Y^{\star}_j \leq 0$) and (2.10), respectively. Further if $j-1 \not> \tau^{l_j-1}$ we have to distinguish between $j = 0 \wedge l_0 = 1$ and $j = \tau^{l_j-1} + 1 \wedge l_j > 1.$ In both cases (2.9) is trivially fulfilled, while (2.10) is void in the first case, and in the second case it reads,

$$0 \leq Z_{\tau^{l_j-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_j-1}}}[Y^{\star}_{\tau^{l_j-1}+1}] + \mathcal{S}_{\tau^{l_j-1}} - \mathcal{S}_{\tau^{l_j-1}+1}, \quad l_j > 1,$$

which is implied by (2.11) and (2.14) for $i = j-1 = \tau^{l_i} = \tau^{l_{j-1}} = \tau^{l_j-1}$. The converse direction, that is from (2.9) and (2.10) to (2.12), (2.13), (2.14), goes similarly and is left to the reader. □

Corollary 2.6. *By Corollary 2.5 there always exists an adapted process* $\mathcal{S}$ *satisfying (2.12), (2.13), (2.14) with* $\mathsf{E}_i\left[\zeta_{i+1}\right] = 0$ *for* $0 \leq i < J$ *due to (2.9) and (2.10). Hence, there exist martingales* $\mathcal{S}$ *that satisfy Lemma 2.4. By Lemma 2.3, for any such martingale* $\mathcal{S}$, $M = M^{\star} - \mathcal{S} \in \mathcal{M}^{\circ}$, *that is,* M *is the optimal martingale.*

Interestingly, the converse to Corollary 2.6 is also true and we so have the following characterization theorem.

Theorem 2.7. *It holds that* $M \in \mathcal{M}^{\circ}$ *if and only if* $M = M^{\star} - \mathcal{S}$, *where* $\mathcal{S}$ *is a martingale with* $\mathcal{S}_0 = 0$ *that satisfies (2.9) and (2.10) in Lemma 2.4.*

The proofs of Lemmas 2.3-2.4 and Theorem 2.7 are given in Section 4. In fact, Theorem 2.7 reveals that, besides the Doob martingale, there generally exists a large set of optimal martingales $M \in \mathcal{M}^{\circ}$. From Theorem 2.7 we also obtain a characterization of the *surely* optimal martingales which is essentially the older result in [13], Thm. 6 (see Section 4 for the proof).

Corollary 2.8. *It holds that* $M \in \mathcal{M}^{\circ\circ}$ *if and only if* $M = M^{\star} - \mathcal{S}$ *with* $\mathcal{S}$ *represented by (2.11) with all* $\mathsf{E}_{\mathcal{F}_i}\left[\zeta_{i+1}\right] = 0$, ζ_{i+1} *satisfying (2.14) for* $i = \tau^{l_i}$, *and* $\zeta_{i+1} = 0$ *for* $\tau^{l_i-1} < i < \tau^{l_i}$, $l_i \geq 1.$

In applications of dual optimal stopping, hence dual martingale minimization, it is usually enough to find martingales M that are "close to" surely optimal ones, merely at some specific point in time i, that is, $M \in \mathcal{M}^{\circ\circ,i}$. Naturally, since $\mathcal{M}^{\circ,i} \supset \mathcal{M}^{\circ}$, we may expect that in general the family of undesirable (not surely) optimal martingales at a specific time may be even much larger than the family $\mathcal{M}^{\circ}$ characterized by Theorem 2.7. A characterization of $\mathcal{M}^{\circ,i}$ and $\mathcal{M}^{\circ\circ,i}$ is given by the next theorem, where we take $i = 0$ without loss of generality. The proof is given in Section 4.

Theorem 2.9. *The following statements hold.*

(i) $M = M^{\star} - \mathcal{S} \in \mathcal{M}^{\circ,0}$ *for some martingale* $\mathcal{S}$ *represented by (2.11), if and only if*

$$\max_{0\leq r<j} \left(Z_r - Y^{*}_r - \mathcal{S}_j + \mathcal{S}_r\right) \leq 0 \quad \textit{for} \quad 0 \leq j \leq \tau^{\star} \quad \textit{and} \tag{2.15}$$

$$\mathcal{S}_j - \mathcal{S}_{\tau^{\star}} \leq Y^{\star}_j - Z_j + A^{*}_j \quad \textit{for} \quad \tau^{\star} < j \leq J, \tag{2.16}$$

where $A^{*}_j = 0$ *(see (2.5)) for all* $0 \leq j \leq \tau^{\star}$.

(ii) $M = M^\star - \mathcal{S} \in \mathcal{M}^{\circ\circ,0}$, *if and only if*

$$\mathcal{S}_j = 0 \quad \text{for} \quad 0 \le j \le \tau^\star, \tag{2.17}$$
$$\mathcal{S}_j \le Y_j^\star - Z_j + A_j^* \quad \text{for} \quad \tau^\star < j \le J. \tag{2.18}$$

After dropping the nonnegative term $Y_j^\star - Z_j$ in the right-hand-sides of (2.16) and (2.18) we may obtain tractable sufficient conditions for a martingale to be optimal or surely optimal at a single date, respectively. In the spirit of Corollary 2.5 they may be formulated in the following way.

Corollary 2.10. *Let $M = M^\star - \mathcal{S}$ for some martingale $\mathcal{S}$ represented by (2.11), then*

(i) $M \in \mathcal{M}^{\circ,0}$ *if*

$$\zeta_j \ \ge \max_{0 \le r < j} \left(Z_r - Y_r^* - \mathcal{S}_{j-1} + \mathcal{S}_r \right) \quad \text{for} \quad 1 \le j \le \tau^\star \quad \text{and}$$
$$\zeta_j \le A_j^* + \mathcal{S}_{\tau^\star} - \mathcal{S}_{j-1} \quad \text{for} \quad \tau^\star < j \le J, \tag{2.19}$$

(ii) $M \in \mathcal{M}^{\circ\circ,0}$ *if* $\zeta_j = 0$ *for* $0 \le j \le \tau^\star$, *and*

$$\zeta_j \le A_j^* - \mathcal{S}_{j-1} \quad \text{for} \quad \tau^\star < j \le J. \tag{2.20}$$

In particular, the right-hand-sides in (2.19) and (2.20) are $\mathcal{F}_{j-1}$-measurable.

Remark 2.11. *While the class of optimal martingales $\mathcal{M}^{\circ,0}$ may be quite large in general, it is still possible that it is just a singleton (containing the Doob martingale only). For example, let the cash-flow $Z \ge 0$ be a martingale itself, then it is easy to see that the only optimal martingale (at 0) is $M = M^\star = Z - Z_0$ (the proof is left as an easy exercise).*

3 Randomized dual martingale representations

Let $(\Omega_0, \mathcal{B})$ be some auxiliary measurable space that is "rich enough". Let us consider random variables on $\widetilde{\Omega} := \Omega \times \Omega_0$ that are measurable with respect to the σ-field $\widetilde{\mathcal{F}} := \sigma\left\{F \times B : \ F \in \mathcal{F}, \ B \in \mathcal{B}\right\}$. While abusing notation a bit, $\mathcal{F}$ and $\mathcal{F}_j$ are identified with $\sigma\left\{F \times \Omega_0 : F \in \mathcal{F}\right\} \subset \widetilde{\mathcal{F}}$ and $\sigma\left\{F \times \Omega_0 : F \in \mathcal{F}_j\right\} \subset \widetilde{\mathcal{F}}$, respectively. Let further P be the given "primary" measure on $(\Omega, \mathcal{F})$, and $\widetilde{\mathsf{P}}$ be an extension of P to $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ in the sense that

$$\widetilde{\mathsf{P}}\left(\Omega_0 \times F\right) = \mathsf{P}\left(F\right) \quad \text{for all} \quad F \in \mathcal{F}.$$

In particular, if $X : \widetilde{\Omega} \to \mathbb{R}$ is $\mathcal{F}$-measurable, then $\left\{(\omega, \omega_0) : X\left(\omega, \omega_0\right) \le x\right\} = \left\{(\omega, \omega_0) : \omega \in F_x\right\}$ for some $F_x \in \mathcal{F}$, that is, X does not depend on ω_0. We now introduce randomized or "pseudo" martingales as random perturbations of $\mathcal{F}$-adapted martingales of the form (2.11). Let $(\eta_j)_{j \ge 0}$ be random variables on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathsf{P}})$ such that $\widetilde{\mathsf{E}}_{\mathcal{F}}\left[\eta_j\right] = 0$ for $j = 0, \ldots, J$. Then

$$\widetilde{M}_j := M_j - \eta_j = M_j^\star - \mathcal{S}_j - \eta_j \tag{3.1}$$

is said to be a pseudo martingale. As such, $\widetilde{M}$ is not an $\mathcal{F}$-martingale but $\widetilde{\mathsf{E}}_{\mathcal{F}}[\widetilde{M}]$ is. The results below on pseudo-martingales provide the key motivation for randomized dual optimal stopping. All proofs in this section are deferred to Section 4.

Proposition 3.1. *For any $\widetilde{M}$ of the form (3.1) one has the upper estimate*

$$\widetilde{\mathsf{E}}\Big[\max_{0\le j\le J}(Z_j-\widetilde{M}_j)\Big]\ge Y_0^\star. \tag{3.2}$$

If $\mathcal{S}=0$, that is,

$$\widetilde{M}_j=M_j^\star-\eta_j \tag{3.3}$$

and the random perturbations (η_j) satisfy in addition

$$\eta_j\le Y_j^\star-Z_j+A_j^\star,\quad \widetilde{\mathsf{P}}-a.s.\quad j=0,\dots,J, \tag{3.4}$$

with $(A_j^\star)$ defined in (2.5)*, then one has the almost sure identity*

$$Y_0^\star=\max_{0\le j\le J}(Z_j-\widetilde{M}_j)\quad \widetilde{\mathsf{P}}\text{-a.s.} \tag{3.5}$$

Moreover, for the first optimal stopping time $\tau^\star:=\tau_0^\star$ (see (2.8)) one must have that $\eta_{\tau^\star}=0$ a.s., and if $\tau^\star$ is strict in the sense that

$$Y_{\tau^\star}^\star-\mathsf{E}_{\mathcal{F}_{\tau^\star}}\left[Y_{\tau^\star+1}^\star\right]>0,$$

then $j=\tau^\star$ is the only time j where $\eta_j=0$.

Due to the following theorem, any (weakly or surely) optimal non Doob martingale turns to a non optimal one in the sense that

$$\widetilde{\mathsf{E}}\Big[\max_{0\le j\le J}(Z_j-\widetilde{M}_j)\Big]>Y_0^\star \tag{3.6}$$

after a particular "optimal" randomization.

Theorem 3.2. *Suppose that $M\in\mathcal{M}^{\circ,0}$ and let (η_j) be a sequence of random variables as in Proposition 3.1, given by*

$$\eta_j=\xi_j\left(Y_j^\star-Z_j+A_j^\star\right),\quad 0\le j\le J, \tag{3.7}$$

where the (ξ_j) are assumed to be i.i.d. distributed on $(-\infty,1]$, independent of $\mathcal{F}$ with $\widetilde{\mathsf{E}}\,[\xi_j]=0$. It is further assumed that the r.v. (ξ_j) have a joint continuous density p supported on $(-\infty,1]$ with $p(1)>0$. As such the randomizers (3.7) satisfy (3.4), and Proposition 3.1 thus provides an upper bound (3.2) due to the pseudo martingale $\widetilde{M}=M-\eta$. Now, for the randomized martingale $\widetilde{M}$ one has (3.6) if $M\neq M^\star$ with positive probability.

The following corollary states that an optimally randomized non Doob martingale in $\mathcal{M}^{\circ,0}$, which is thus suboptimal in the sense of (3.6) due to the previous theorem, cannot have zero variance. The proof relies on Theorem 3.2.

Corollary 3.3. *Let $M\in\mathcal{M}^{\circ,0}$, (η_j) as in Theorem 3.2, and $\widetilde{M}=M-\eta$. Then $\operatorname{Var}\big(\max_{0\le j\le J}(Z_j-\widetilde{M}_j)\big)=0$ if and only if $M=M^\star$.*

Discussion

Proposition 3.1 provides us with a remarkable freedom of perturbing the Doob martingale randomly while (3.5) remains true. The bottom line of Theorem 3.2 is that randomization under condition (3.4) of an optimal, or even surely optimal, but *non*-Doob martingale results in a non optimal (pseudo) martingale, while any randomization of the Doob martingale under (3.4) remains a surely optimal pseudo martingale. This is an important feature, since in this way martingale candidates that are optimal but not equal to the (surely optimal) Doob martingale can be sorted out by randomization.

4 Proofs

4.1 Proof of Lemma 2.1

It is enough to show the convexity of $\mathcal{M}^{\circ,j}$ and $\mathcal{M}^{\circ\circ,j}$ for any j. For any $M, M' \in \mathcal{M}^{\circ,j}$ and $\theta \in (0,1)$ one has

$$\begin{aligned}
&\mathsf{E}_{\mathcal{F}_j}\left[\max_{j\le r\le J}(Z_r - (\theta M_r + (1-\theta)M_r')) + \theta M_j + (1-\theta)M_j'\right] \\
&= \mathsf{E}\left[\max_{j\le r\le J}\left(\theta\,(Z_r - M_r + M_j) + (1-\theta)\left(Z_r - M_r' + M_j'\right)\right)\right] \\
&\le \theta\mathsf{E}\left[\max_{j\le r\le J}(Z_r - M_r + M_j)\right] + (1-\theta)\mathsf{E}\left[\max_{j\le r\le J}\left(Z_r - M_r' + M_j'\right)\right] = Y_j^\star
\end{aligned}$$

while by (2.2),

$$\mathsf{E}_{\mathcal{F}_j}\left[\max_{j\le r\le J}\left(Z_r - (\theta M_r + (1-\theta)M_r') + \theta M_j + (1-\theta)M_j'\right)\right] \ge Y_j^\star.$$

Similarly, for any $M, M' \in \mathcal{M}^{\circ\circ,j}$ and $\theta \in (0,1)$ we have

$$\begin{aligned}
&\max_{j\le r\le J}\left(Z_r - \left(\theta M_r + (1-\theta)M_r' + \theta M_j + (1-\theta)M_j'\right)\right) \\
&= \max_{j\le r\le J}\left(\theta\,(Z_r - M_r + M_j) + (1-\theta)\left(Z_r - M_r' + M_j'\right)\right) \\
&\le \theta \max_{j\le r\le J}(Z_r - M_r + M_j) + (1-\theta)\max_{0\le r\le J}\left(Z_r - M_r' + M_j'\right) = Y_j^\star
\end{aligned}$$

while by (2.2),

$$\mathsf{E}_{\mathcal{F}_j}\left[\max_{j\le r\le J}\left(Z_r - (\theta M_r + (1-\theta)M_r') + \theta M_j + (1-\theta)M_j'\right)\right] \ge Y_j^\star.$$

In both cases the sandwich property completes.

4.2 Proof of Lemma 2.3

Suppose that M is a martingale with $M_0 = 0$ such that Lemma 2.3-(i) and (ii) hold. Then (ii) implies for $q \ge 1$ that

$$Z_{\tau^1} - M_{\tau^1} \ge Z_{\tau^2} - M_{\tau^2} \ge \ldots \ge Z_{\tau^q} - M_{\tau^q} \tag{4.1}$$

Now take $0 \le i \le J$ arbitrarily, and let $q_i \ge 1$ be such that $\tau^{q_i - 1} < i \le \tau^{q_i}$ (Note that q_i is unique and $\mathcal{F}_i$ measurable). Then due to Lemma 2.3-(i) and (4.1),

$$\begin{aligned}
\max_{i\le r\le J}(Z_r - M_r) &= \max\left(\max_{i\le r\le \tau^{q_i}}(Z_r - M_r), \max_{q>q_i}\max_{\tau^{q-1}<r\le\tau^q}(Z_r - M_r)\right) \\
&= \max\left(Z_{\tau^{q_i}} - M_{\tau^{q_i}}, \max_{q>q_i}(Z_{\tau^q} - M_{\tau^q})\right) \\
&= \max\left(Z_{\tau^{q_i}} - M_{\tau^{q_i}}, Z_{\tau^{q_i+1}} - M_{\tau^{q_i+1}}\right) = Z_{\tau^{q_i}} - M_{\tau^{q_i}}.
\end{aligned}$$

On the other hand, one has $\tau_i^\star = \tau^{q_i}$ (see (2.8)). Thus, by Proposition 2.2, $M \in \mathcal{M}^{\circ,i}$ and hence $M \in \mathcal{M}^\circ$ since i was taken arbitrarily.

Conversely, suppose that $M \in \mathcal{M}^\circ$. So for any $0 \le i \le J$,

$$\max_{i\le r\le J}(Z_r - M_r) = Z_{\tau_i^\star} - M_{\tau_i^\star}$$

by Proposition 2.2. For $l = 1$ one thus has

$$\max_{\tau^0<r\le J}(Z_r - M_r) = \max_{0\le r\le J}(Z_r - M_r) = Z_{\tau_0^\star} - M_{\tau_0^\star} = Z_{\tau^1} - M_{\tau^1}$$

and for $l > 1$ it holds that

$$\begin{aligned}\max_{\tau^{l-1}<r\le J}(Z_r - M_r) &= \sum_{k=0}^{J-1} 1_{\{\tau^{l-1}=k\}} \max_{k+1\le r\le J}(Z_r - M_r) \\ &= \sum_{k=0}^{J-1} 1_{\{\tau^{l-1}=k\}} \left(Z_{\tau_{k+1}^\star} - M_{\tau_{k+1}^\star}\right) \\ &= \sum_{k=0}^{J-1} 1_{\{\tau^{l-1}=k\}} \left(Z_{\tau^l} - M_{\tau^l}\right) = Z_{\tau^l} - M_{\tau^l}.\end{aligned}$$

That is, (i) is shown. Next, for any $l > 1$ it holds

$$\begin{aligned}\max_{\tau^{l-1}\le r\le J}(Z_r - M_r) &= \sum_{k=0}^{L} 1_{\{\tau^{l-1}=k\}} \max_{k\le r\le J}(Z_r - M_r) \\ &= \sum_{k=0}^{L} 1_{\{\tau^{l-1}=k\}} \left(Z_{\tau_k^\star} - M_{\tau_k^\star}\right) \\ &= Z_{\tau^\star_{\tau^{l-1}}} - M_{\tau^\star_{\tau^{l-1}}} = Z_{\tau^{l-1}} - M_{\tau^{l-1}}\end{aligned}$$

which implies (ii).

4.3 Proof of Lemma 2.4

Assume that $\mathcal{S}$ is adapted with $\mathcal{S}_0 = 0$ and that $\mathcal{S}$ satisfies (2.9) and (2.10). For $l > 1$ and $\tau^{l-1} < r \le \tau^l$ we may write,

$$\begin{aligned}Z_r - M_r &= Z_r - M_r^\star + \mathcal{S}_r \qquad (4.2) \\ &= Z_r - M_{\tau^{l-1}}^\star + M_{\tau^{l-1}}^\star - M_r^\star + \mathcal{S}_r \\ &= Z_r - M_{\tau^{l-1}}^\star + \mathcal{S}_r - \sum_{k=\tau^{l-1}+1}^{r} \left(Y_k^\star - \mathsf{E}_{\mathcal{F}_{k-1}}\left[Y_k^\star\right]\right) \\ &= Z_r - M_{\tau^{l-1}}^\star + \mathcal{S}_r \\ &\quad - \sum_{k=\tau^{l-1}+1}^{r} Y_k^\star + \sum_{k=\tau^{l-1}+1}^{r-1} \mathsf{E}_{\mathcal{F}_k}\left[Y_{k+1}^\star\right] + \mathsf{E}_{\mathcal{F}_{\tau^{l-1}}}\left[Y_{\tau^{l-1}+1}^\star\right] \\ &= Z_r - Y_r^\star - M_{\tau^{l-1}}^\star + \mathsf{E}_{\mathcal{F}_{\tau^{l-1}}}\left[Y_{\tau^{l-1}+1}^\star\right] + \mathcal{S}_r.\end{aligned}$$

By taking $r = \tau^l$ in (4.2) and using $Z_{\tau^l} = Y^\star_{\tau^l}$ we then get

$$Z_{\tau^l} - M_{\tau^l} = -M^\star_{\tau^{l-1}} + \mathsf{E}_{\mathcal{F}_{\tau^{l-1}}}\left[Y^\star_{\tau^{l-1}+1}\right] + \mathcal{S}_{\tau^l}$$

and thus

$$Z_r - M_r = Z_{\tau^l} - M_{\tau^l} + Z_r - Y^\star_r + \mathcal{S}_r - \mathcal{S}_{\tau^l}, \quad \tau^{l-1} < r \le \tau^l.$$

So from (2.9) we obtain with $i = \tau^l$, $l_i - 1 = l - 1$,

$$Z_r - M_r \le Z_{\tau^l} - M_{\tau^l} \quad \text{for} \quad \tau^{l-1} < r \le \tau^l,$$

i.e. Lemma 2.3-(i) for $l > 1$. If $l = 1$ and $\tau^1 = 0$, Lemma 2.3-(i) is trivially fulfilled. So let us consider $l = 1$ and $\tau^1 > 0$. Analogously, we then may write for $\tau^0 = 0^- < 0 < r \le \tau^1$,

$$\begin{aligned}
Z_r - M_r = Z_r - M^\star_r + \mathcal{S}_r &= Z_r + \mathcal{S}_r - \sum_{k=1}^{r}\left(Y^\star_k - \mathsf{E}_{\mathcal{F}_{k-1}}\left[Y^\star_k\right]\right) \\
= Z_r + \mathcal{S}_r - \sum_{k=1}^{r} Y^\star_k &+ \sum_{k=1}^{r-1}\mathsf{E}_{\mathcal{F}_k}\left[Y^\star_{k+1}\right] + \mathsf{E}_{\mathcal{F}_{\tau^{l-1}}}\left[Y^\star_{\tau^{l-1}+1}\right] \\
&= Z_r - Y^\star_r + \mathsf{E}_{\mathcal{F}_0}\left[Y^\star_1\right] + \mathcal{S}_r.
\end{aligned} \tag{4.3}$$

It is easy to see that (4.3) is also valid for $r = 0$, due to our assumption $\tau^1 > 0$. Thus, for $l = 1$ and taking $r = \tau^1 > 0$, we get from (4.3),

$$Z_{\tau^1} - M_{\tau^1} = \mathsf{E}_{\mathcal{F}_0}\left[Y^\star_1\right] + \mathcal{S}_{\tau^1},$$

whence (4.3) implies for $\tau^0 = 0^- < r \le \tau^1$

$$Z_r - M_r = Z_r - Y^\star_r + Z_{\tau^1} - M_{\tau^1} \le Z_{\tau^1} - M_{\tau^1},$$

that is Lemma 2.3-(i) holds also for $l = 1$.

Let us now consider (ii) and take $l > 1$. Now for $\tau^{l-1} < r \le \tau^l$ (4.2) implies with $M^\star_{\tau^{l-1}} = \mathcal{S}_{\tau^{l-1}} + M_{\tau^{l-1}}$,

$$Z_r - M_r = Z_{\tau^{l-1}} - M_{\tau^{l-1}} + Z_r - Y^\star_r + \mathsf{E}_{\mathcal{F}_{\tau^{l-1}}}\left[Y^\star_{\tau^{l-1}+1}\right] - Z_{\tau^{l-1}} + \mathcal{S}_r - \mathcal{S}_{\tau^{l-1}}. \tag{4.4}$$

Hence, since always $Z_r \le Y^\star_r$, (2.10) implies for $\tau^{l-1} < r \le \tau^l$,

$$Z_r - M_r \le Z_{\tau^{l-1}} - M_{\tau^{l-1}}, \quad \tau^{l-1} < r \le \tau^l, \tag{4.5}$$

i.e. Lemma 2.3-(ii) is proved.

4.4 Proof of Theorem 2.7

If $M = M^\star - \mathcal{S}$, where $\mathcal{S}$ is a martingale with $\mathcal{S}_0 = 0$ that satisfies (2.9) and (2.10) in Lemma 2.4 then $M \in \mathcal{M}^\circ$ due to Corollary 2.6.

Let us now consider the converse and assume that $M = M^\star - \mathcal{S} \in \mathcal{M}^\circ$ with $M_0 = \mathcal{S}_0 = 0$. Then $\mathcal{S}$ is adapted and may be written in the form (2.11) where the ζ_{i+1} are $\mathcal{F}_{i+1}$-measurable and $\mathsf{E}_{\mathcal{F}_i}\left[\zeta_{i+1}\right] = 0$ for $0 \le i < J$. Since $M \in \mathcal{M}^\circ$ Lemma 2.3-(i) implies that for $l \ge 1$,

$$\max_{\tau^{l-1} < r \le \tau^l}\left(Z_r - Z_{\tau^l} + M^\star_{\tau^l} - M^\star_r + \mathcal{S}_r - \mathcal{S}_{\tau^l}\right) = 0, \quad \text{hence}$$

$$\max_{\tau^{l-1} < r \le \tau^l}\left(Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_{\tau^l}\right) = 0 \tag{4.6}$$

since for each r with $\tau^{l-1} < r \leq \tau^l$ one has $Z_{\tau^l} - M^\star_{\tau^l} + M^\star_r = Z_{\tau^\star_r} - M^\star_{\tau^\star_r} + M^\star_r = Y^*_r$ because $M^\star \in \mathcal{M}^{\circ\circ}$. We now show for any i with $\tau^{l-1} < i \leq \tau^l$ that (2.9) holds with $l_i = l$ by backward induction. For $i = \tau^{l_i}$ it follows from (4.6). Now suppose that for some i with $\tau^{l_i-1} < i < i+1 \leq \tau^{l_i}$ it holds that

$$1_{\left\{\tau^{l_{i+1}-1}<i+1\leq\tau^{l_{i+1}}\right\}} \max_{\tau^{l_{i+1}-1}<r\leq i+1} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_{i+1}) \leq 0. \tag{4.7}$$

One has by construction

$$\max_{\tau^{l_i-1}<r\leq i} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_i) = \zeta_{i+1} + \max_{\tau^{l_i-1}<r\leq i} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_{i+1}).$$

Hence, since $\left\{\tau^{l_i-1} < i < \tau^{l_i}\right\} = \left\{\tau^{l_i-1} < i\right\} \cap \left\{\tau^{l_i-1} < i+1 \leq \tau^{l_i}\right\}$ with $\left\{\tau^{l_i-1} < i\right\} \in \mathcal{F}_i$ and $\left\{\tau^{l_i-1} < i+1 \leq \tau^l\right\} \in \mathcal{F}_i$ (!), $\mathsf{E}_{\mathcal{F}_i}\left[\zeta_{i+1}\right] = 0$, $l_i = l_{i+1}$, and taking $\mathcal{F}_i$-conditional expectations,

$$\begin{aligned}
&1_{\left\{\tau^{l_i-1}<i<\tau^{l_i}\right\}} \max_{\tau^{l-1}<r\leq i} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_i) \\
&\quad = 1_{\left\{\tau^{l_i-1}<i\right\}} \mathsf{E}_{\mathcal{F}_i}\left[\max_{\tau^{l_i-1}<r\leq i} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_{i+1}) 1_{\left\{\tau^{l_i-1}<i+1\leq\tau^{l_i}\right\}}\right] \\
&\quad \leq 1_{\left\{\tau^{l_i-1}<i\right\}} \mathsf{E}_{\mathcal{F}_i}\left[\max_{\tau^{l_{i+1}-1}<r\leq i+1} (Z_r - Y^*_r + \mathcal{S}_r - \mathcal{S}_{i+1}) 1_{\left\{\tau^{l_{i+1}-1}<i+1\leq\tau^{l_{i+1}}\right\}}\right] \leq 0,
\end{aligned}$$

using the induction hypothesis (4.7). In view of (4.6) it follows that (2.9) holds for $\tau^{l_i-1} < i \leq \tau^{l_i}$.

Next, on the other hand, $M \in \mathcal{M}^\circ$ implies by Lemma 2.3-(ii) that for any fixed $l > 1$,

$$\max_{\tau^{l-1}\leq r\leq\tau^l} (Z_r - M^\star_r + \mathcal{S}_r) = Z_{\tau^{l-1}} - M^\star_{\tau^{l-1}} + \mathcal{S}_{\tau^{l-1}}, \quad \text{hence}$$

$$\max_{\tau^{l-1}<r\leq\tau^l} (Z_r - Z_{\tau^{l-1}} + M^\star_{\tau^{l-1}} - M^\star_r + \mathcal{S}_r - \mathcal{S}_{\tau^{l-1}}) = 0. \tag{4.8}$$

Suppose that $\tau^{l-1} < i \leq \tau^l$ and hence $l_i = l$. Then (4.8) implies by (2.11) after a few manipulations,

$$\begin{aligned}
&Z_i - Z_{\tau^{l_i-1}} + M^\star_{\tau^{l_i-1}} - M^\star_i + \mathcal{S}_i - \mathcal{S}_{\tau^{l_i-1}} \\
&= \zeta_{\tau^{l_i-1}+1} + \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^\star_{\tau^{l_i-1}+1}\right] - Z_{\tau^{l_i-1}} + Z_i - Y^\star_i \\
&+ \sum_{r=\tau^{l_i-1}+1}^{i-1} \zeta_{r+1} + \sum_{r=\tau^{l_i-1}+1}^{i-1} \mathsf{E}_{\mathcal{F}_r}\left[Y^\star_{r+1}\right] - \sum_{r=\tau^{l_i-1}+1}^{i-1} Y^\star_r \leq 0
\end{aligned}$$

with the usual convention $\sum_{r=p}^{p-1} := 0$. Thus, either the last three sums are zero due to $i = \tau^{l_i-1} + 1$, or we may use that $Y^\star_r = \mathsf{E}_{\mathcal{F}_r}\left[Y^\star_{r+1}\right]$ for $\tau^{l_i-1} < r < i$. We thus get for $\tau^{l-1} < i \leq \tau^l$,

$$\zeta_{\tau^{l_i-1}+1} + \mathsf{E}_{\tau^{l_i-1}}\left[Y^\star_{\tau^{l_i-1}+1}\right] - Z_{\tau^{l_i-1}} + Z_i - Y^\star_i + \mathcal{S}_i - \mathcal{S}_{\tau^{l_i-1}+1} \leq 0. \tag{4.9}$$

In particular, due to $Z_{\tau^l} = Y^\star_{\tau^l}$, for $i = \tau^l$ this gives

$$\zeta_{\tau^{l_i-1}+1} + \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^\star_{\tau^{l_i-1}+1}\right] - Z_{\tau^{l_i-1}} + \mathcal{S}_{\tau^{l_i}} - \mathcal{S}_{\tau^{l_i-1}+1} \leq 0. \tag{4.10}$$

Let us now show that (2.10) holds for $\tau^{l_i-1} < i \leq \tau^{l_i}$ and $l_i > 1$ by backward induction. For $i = \tau^{l_i}$ it follows from (4.10) by $\zeta_{\tau^{l_i-1}+1} - \mathcal{S}_{\tau^{l_i-1}+1} = -\mathcal{S}_{\tau^{l_i-1}}$ that

$$Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^\star_{\tau^{l_i-1}+1}\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_{\tau^{l_i}} \geq 0$$

that is (2.10) for $i = \tau^{l_i}$. Now suppose that for some i with $\tau^{l_i-1} < i < i+1 \le \tau^{l_i}$ it holds that

$$1_{\left\{\tau^{l_{i+1}-1}<i+1\le\tau^{l_{i+1}}\right\}}\left(Z_{\tau^{l_{i+1}-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_{i+1}-1}}}\left[Y^{\star}_{\tau^{l_{i+1}-1}+1}\right] + \mathcal{S}_{\tau^{l_{i+1}-1}} - \mathcal{S}_{i+1}\right) \ge 0.$$

One thus has by construction

$$\begin{aligned}&Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^{\star}_{\tau^{l_i-1}+1}\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_i \\ &\qquad = Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^{\star}_{\tau^{l_i-1}+1}\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_{i+1} + \zeta^{-}_{i+1}.\end{aligned}$$

It then follows similarly by taking $\mathcal{F}_i$-conditional expectations that

$$\begin{aligned}&1_{\left\{\tau^{l_i-1}<i<\tau^{l_i}\right\}}\left(Z_{\tau^{l_i-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_i-1}}}\left[Y^{\star}_{\tau^{l_i-1}+1}\right] + \mathcal{S}_{\tau^{l_i-1}} - \mathcal{S}_i\right) = 1_{\left\{\tau^{l_i-1}<i<\tau^{l_i}\right\}} \\ &\quad \times \mathsf{E}_{\mathcal{F}_i}\left[\left(Z_{\tau^{l_{i+1}-1}} - \mathsf{E}_{\mathcal{F}_{\tau^{l_{i+1}-1}}}\left[Y^{\star}_{\tau^{l_{i+1}-1}+1}\right] + \mathcal{S}_{\tau^{l_{i+1}-1}} - \mathcal{S}_{i+1}\right) 1_{\left\{\tau^{l_{i+1}-1}<i+1\le\tau^{l_{i+1}}\right\}}\right] \ge 0\end{aligned}$$

by the induction hypothesis (note again that $l_{i+1} = l_i$). Thus, (2.10) holds for $\tau^{l_i-1} < i \le \tau^{l_i}$ and so (2.10) is proved. We thus conclude that $\mathcal{S}$ is a martingale that satisfies (2.9) and (2.10). The theorem is proved.

4.5 Proof of Corollary 2.8

Suppose that $M = M^{\star} - \mathcal{S} \in \mathcal{M}^{\circ\circ}$ for some martingale $\mathcal{S}$ represented by (2.11). Since $M \in \mathcal{M}^{\circ\circ} \subset \mathcal{M}^{\circ}$, Theorem 2.7 implies (via Corollary 2.5) that the ζ_{i+1} satisfy (2.14) for $i = \tau^{l_i}$. Further, for any $0 \le i \le J$ one has

$$\begin{aligned}Y^{\star}_i &= \max_{i\le r\le J}\left(Z_r - M_r + M_i\right) = \max_{i\le r\le J}\left(Z_r - M^{\star}_r + M^{\star}_i + \mathcal{S}_r - \mathcal{S}_i\right) \\ &\le Z_{\tau^{\star}_i} - M^{\star}_{\tau^{\star}_i} + M^{\star}_i + \mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i = Y^{\star}_i + \mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i\end{aligned}$$

since $M^{\star} \in \mathcal{M}^{\circ\circ}$. So

$$\mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i \ge 0 \quad \text{while} \quad \mathsf{E}_{\mathcal{F}_i}\left[\mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i\right] = 0,$$

by Doob's sampling theorem. Hence, by the sandwich property, $\mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i = 0$ for all $0 \le i \le J$. This implies for any i with $\tau^{l-1} < i < \tau^{l}$ that

$$\zeta_{i+1} = \mathcal{S}_{i+1} - \mathcal{S}_i = \mathcal{S}_{\tau^{\star}_{i+1}} - \mathcal{S}_{\tau^{\star}_i} = 0$$

due to $\tau^{\star}_i = \tau^{\star}_{i+1} = \tau^{l}$.

Conversely, if the ζ_{i+1} satisfy (2.14) for $i = \tau^{l_i}$ and further $\zeta_{i+1} = 0$ for any i with $\tau^{l_{i-1}} < i < \tau^{l_i} = \tau^{\star}_i$, they also trivially satisfy (2.13) and (2.12), and so one has $M \in \mathcal{M}^{\circ}$ by Theorem 2.7 (via Corollary 2.5). Furthermore it follows that $\mathcal{S}_{\tau^{\star}_i} = \mathcal{S}_i$ for any i with $\tau^{l_{i-1}} < i < \tau^{l_i} = \tau^{\star}_i$, so by Proposition 2.2

$$\begin{aligned}\max_{i\le r\le J}\left(Z_r - M_r\right) &= Z_{\tau^{\star}_i} - M_{\tau^{\star}_i} = Z_{\tau^{\star}_i} - M^{\star}_{\tau^{\star}_i} + \mathcal{S}_{\tau^{\star}_i} \\ &= Y^{\star}_i - M^{\star}_i + \mathcal{S}_{\tau^{\star}_i} = Y^{\star}_i - M_i + \mathcal{S}_{\tau^{\star}_i} - \mathcal{S}_i \\ &= Y^{\star}_i - M_i.\end{aligned}$$

Hence, $M \in \mathcal{M}^{\circ\circ,i}$ and so $M \in \mathcal{M}^{\circ\circ}$ since i was arbitrary.

4.6 Proof of Theorem 2.9

(i): Due to Proposition 2.2, $M \in \mathcal{M}^{\circ,0}$ if and only if

$$0 = \max_{0 \le r \le J} \left(Z_r - M_r - Z_{\tau^\star} + M_{\tau^\star} \right)$$

with $\tau^\star := \tau_0^\star$, which is equivalent with

$$\max_{0 \le r < \tau^\star} \left(Z_r - M_r - Z_{\tau^\star} + M_{\tau^\star} \right) \le 0 \quad \text{and} \tag{4.11}$$

$$\max_{\tau^\star < r \le J} \left(Z_r - M_r - Z_{\tau^\star} + M_{\tau^\star} \right) \le 0. \tag{4.12}$$

Since $\tau^\star = \tau_r^\star$ for $0 \le r < \tau^\star$, (4.11) reads

$$\begin{aligned} \max_{0 \le r < \tau^\star} \left(Z_r - M_r^\star - Z_{\tau_r^\star} + M^*_{\tau_r^\star} - \mathcal{S}_{\tau_r^\star} + \mathcal{S}_r \right) &= \max_{0 \le r < \tau^\star} \left(Z_r - Y_r^* - \mathcal{S}_{\tau_r^\star} + \mathcal{S}_r \right) \\ &= \max_{0 \le r < \tau^\star} \left(Z_r - Y_r^* - \mathcal{S}_{\tau^\star} + \mathcal{S}_r \right) \le 0 \end{aligned} \tag{4.13}$$

which in turn is equivalent with (2.15). Indeed, suppose that (4.13) holds. Then (2.15) clearly holds for $j = \tau^\star$. Now assume that (2.15) holds for $0 < j \le \tau^\star$. Then, by backward induction,

$$\max_{0 \le r < j-1} \left(Z_r - Y_r^* - \mathcal{S}_{j-1} + \mathcal{S}_r \right) = \max_{0 \le r < j-1} \left(Z_r - Y_r^* - \mathcal{S}_j + \mathcal{S}_r \right) + \zeta_j \le \zeta_j$$

By next taking $\mathcal{F}_{j-1}$-conditional expectations we get (2.15) for $j-1$. For the converse, just take $j = \tau^\star$ in (2.15). We next consider (4.12), which may be written as

$$\max_{\tau^\star < r \le J} \left(Z_r - M_r^* + M^*_{\tau^\star} - Z_{\tau^\star} - \mathcal{S}_{\tau^\star} + \mathcal{S}_r \right) \le 0$$

Using the Doob decomposition of the Snell envelope (2.5), $A^*_{\tau^\star} = 0$, and that $Y^\star_{\tau^\star} = Z_{\tau^\star}$, this is equivalent with (2.16).

(ii): Suppose that $M \in \mathcal{M}^{\circ\circ,0}$. One has that $M = M^\star - \mathcal{S} \in \mathcal{M}^{\circ\circ,0}$, if and only if

$$0 = \max_{0 \le r \le J} \left(Z_r - M_r - Y_0^* \right) = \max_{0 \le r \le J} \left(Z_r - M_r^* + \mathcal{S}_r - Y_0^* \right).$$

Since $Z_{\tau^\star} - M^*_{\tau^\star} = Y_0^*$ a.s., this implies $\mathcal{S}_{\tau^\star} \le 0$ a.s., and so by $\mathsf{E}_{\mathcal{F}_0}\left[\mathcal{S}_{\tau^\star}\right] = 0$, that $\mathcal{S}_{\tau^\star} = 0$ by the sandwich property. Now note that $\widetilde{\mathcal{S}}_j = \mathcal{S}_{j \wedge \tau^\star}$, $j = 0, ..., J$, is also a martingale with $\widetilde{\mathcal{S}}_J = 0$ a.s. Let us write (assuming that $J \ge 1$)

$$0 = \widetilde{\mathcal{S}}_J = \sum_{j=1}^{J} \widetilde{\mathcal{S}}_j - \widetilde{\mathcal{S}}_{j-1} = \widetilde{\mathcal{S}}_J - \widetilde{\mathcal{S}}_{J-1} + \sum_{j=1}^{J-1} \widetilde{\mathcal{S}}_j - \widetilde{\mathcal{S}}_{j-1}.$$

That is, $\widetilde{\mathcal{S}}_J - \widetilde{\mathcal{S}}_{J-1}$ is $\mathcal{F}_{J-1}$-measurable with $\mathsf{E}_{\mathcal{F}_{J-1}}\left[\widetilde{\mathcal{S}}_J - \widetilde{\mathcal{S}}_{J-1}\right] = 0$, so $\widetilde{\mathcal{S}}_J - \widetilde{\mathcal{S}}_{J-1} = 0$ and thus $\widetilde{\mathcal{S}}_{J-1} = 0$ a.s. By proceeding backwards in the same way we see that $\widetilde{\mathcal{S}}_j - \widetilde{\mathcal{S}}_{j-1} = 0$ for all $1 \le j \le J$, which implies

$$\widetilde{\mathcal{S}}_j - \widetilde{\mathcal{S}}_{j-1} = \sum_{r=1}^{j \wedge \tau^\star} \zeta_r - \sum_{r=1}^{(j-1) \wedge \tau^\star} \zeta_r = 1_{\{\tau^\star \ge j\}} \zeta_j = 0,$$

whence $\mathcal{S}_j = 0$ for $0 \le j \le \tau^\star$, i.e. (2.17). Since $\mathcal{M}^{\circ\circ,0} \subset \mathcal{M}^{\circ,0}$ (2.18) follows from (2.16) with $\mathcal{S}_{\tau^\star} = 0$. Conversely, if (2.17) and (2.18) hold, then

$$\begin{aligned}\max_{0\le r\le J}(Z_r - M^*_r + \mathcal{S}_r - Y^*_0) &= \max_{0\le r\le \tau^\star}(Z_r - M^*_r - Y^*_0) \vee \max_{\tau^\star<r\le J}(Z_r - M^*_r + \mathcal{S}_r - Y^*_0)\\ &= 0 \vee \max_{\tau^\star<r\le J}(Z_r - M^*_r + \mathcal{S}_r - Y^*_0)\end{aligned}$$

and due to (2.18), for each $\tau^\star < r \le J$

$$Z_r - M^*_r + \mathcal{S}_r - Y^*_0 \le Y^\star_r - M^*_r + A^*_r - Y^*_0 = 0$$

by (2.5). That is $\max_{0\le r\le J}(Z_r - M_r) = Y^*_0$ and so $M \in \mathcal{M}^{\circ\circ,0}$.

4.7 Proof of Proposition 3.1

It holds that

$$\begin{aligned}\widetilde{\mathsf{E}}\left[\max_{0\le j\le J}\left(Z_j - \widetilde{M}_j\right)\right] &= \widetilde{\mathsf{E}}\widetilde{\mathsf{E}}_{\mathcal{F}}\left[\max_{0\le j\le J}\left(Z_j - M^\star_j + \mathcal{S}_j + \eta_j\right)\right]\\ &\ge \widetilde{\mathsf{E}}\left[\max_{0\le j\le J}\left(Z_j - M^\star_j + \mathcal{S}_j + \widetilde{\mathsf{E}}_{\mathcal{F}}[\eta_j]\right)\right]\\ &= \mathsf{E}\left[\max_{0\le j\le J}\left(Z_j - M^\star_j + \mathcal{S}_j\right)\right] \ge Y^\star_0,\end{aligned}$$

by duality, hence (3.2). Further, if $\mathcal{S} = 0$ and (3.4) applies, we may write

$$\begin{aligned}Z_j - \widetilde{M}_j &= Z_j - M^\star_j + \eta_j\\ &= Z_j - \left(Y^\star_j + A^\star_j - Y^\star_0\right) + \eta_j\\ &= Y^\star_0 + Z_j - Y^\star_j - A^\star_j + \eta_j \le Y^\star_0 \end{aligned} \tag{4.14}$$

Then (3.5) follows by (3.2) and the sandwich property.

As for the last statement: If $\tau^\star = 0$ one has $Z_0 = Y^\star_0$ and $A^\star_0 = 0$ by definition, hence in (3.4) $\eta^0 \le 0$ a.s., which implies $\eta_0 = 0$. If $\tau^\star > 0$ one has $Z_{\tau^\star} = Y^\star_{\tau^\star}$ and $A^\star_j - A^\star_{j-1} = Y^\star_{j-1} - \mathsf{E}_{\mathcal{F}_{j-1}}\left[Y^\star_j\right] = 0$ for $j = 1, ..., \tau^\star$, hence $A^\star_{\tau^\star} = 0$ and so $\eta_{\tau^\star} \le 0$ due to (3.4), implying $\eta_{\tau^\star} = 0$. If $\tau^\star$ is strictly optimal, that is $A^\star_{\tau^\star+1} = A^\star_{\tau^\star+1} - A^\star_{\tau^\star} = Y^\star_{\tau^\star} - \mathsf{E}_{\mathcal{F}_{\tau^\star}}\left[Y^\star_{\tau^\star+1}\right] > 0$, one has

$$Y^\star_j - Z_j + A^\star_j > 0 \quad \text{for all } j \ne \tau^\star$$

since always $Y^\star_j \ge Z_j$ and $A^\star_j \ge 0$, $Y^\star_j > Z_j$ for $0 \le j < \tau^\star$, and $A^\star_j \ge A^\star_{\tau^\star+1}$ for $j > \tau^\star$ (remember that A is nondecreasing).

4.8 Proof of Theorem 3.2

Let $M = M^\star - \mathcal{S} \in \mathcal{M}^{\circ,0}$, let (η_j) be as stated, and let us assume that

$$\widetilde{\mathsf{E}}\Big[\max_{0\le j\le J}(Z_j - \widetilde{M}_j)\Big] = Y^\star_0. \tag{4.15}$$

We then have to show that $M = M^\star$. By using (2.5) we may write

$$\begin{aligned}\max_{0\le j\le J}\left(Z_j - \widetilde{M}_j\right) &= \max_{0\le j\le J}\left(Z_j - M_j^\star + \mathcal{S}_j + \eta_j\right)\\ &= Y_0^\star + \max_{0\le j\le J}\left(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star\right).\end{aligned}$$

By (4.15) we must have

$$\widetilde{\mathsf{E}}\left[\max_{0\le j\le J}\left(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star\right)\right] = 0. \tag{4.16}$$

We observe that

$$\max_{0\le j\le J}\left(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star\right) \ge \mathcal{S}_{\tau^\star} + \eta_{\tau^\star} + Z_{\tau^\star} - Y_{\tau^\star}^\star - A_{\tau^\star}^\star = \mathcal{S}_{\tau^\star},$$

using $\eta_{\tau^\star} = 0$ due to Proposition 3.1. By Doob's sampling theorem, $\widetilde{\mathsf{E}}\left[\mathcal{S}_{\tau^\star}\right] = 0$ and so (4.16) implies by the sandwich property,

$$\begin{aligned}\max_{0\le j\le J}\left(\mathcal{S}_j - \mathcal{S}_{\tau^\star} + \eta_j + Z_j - Y_j^\star - A_j^\star\right) = 0, \quad \text{a.s., whence}\\ \eta_j \le \mathcal{S}_{\tau^\star} - \mathcal{S}_j + Y_j^\star - Z_j + A_j^\star \quad \text{a.s. for all } 0\le j\le J.\end{aligned} \tag{4.17}$$

Let us fix some $0 \le j \le J$ and assume that $\mathsf{P}(0 \le j < \tau^\star) > 0$. Due to (3.7) we thus have that,

$$\xi_j 1_{\{0\le j<\tau^\star\}} \le \left(1 + \frac{\mathcal{S}_{\tau^\star} - \mathcal{S}_j}{Y_j^\star - Z_j + A_j^\star}\right) 1_{\{0\le j<\tau^\star\}} \quad \text{almost surely.} \tag{4.18}$$

(note that $A_j^\star \ge 0$ and $Y_j^\star > Z_j$ for $0 \le j < \tau^\star$). Since $M \in \mathcal{M}^{\circ,0}$, $0 \le j < \tau^\star$ implies by (2.15) $\mathcal{S}_{\tau^\star} - \mathcal{S}_j \ge Z_j - Y_j^\star$. Now assume that for some $\epsilon > 0$ but small enough, the set

$$\mathcal{C}_j^\epsilon := \{0 \le j < \tau^\star\} \cap \left\{0 > -\epsilon(Y_j^\star - Z_j + A_j^\star) > \mathcal{S}_{\tau^\star} - \mathcal{S}_j \ge Z_j - Y_j^\star\right\}$$

has positive probability. Since on $\mathcal{C}_j^\epsilon$ one has

$$1 - \frac{Y_j^\star - Z_j}{Y_j^\star - Z_j + A_j^\star} \le 1 + \frac{\mathcal{S}_{\tau^\star} - \mathcal{S}_j}{Y_j^\star - Z_j + A_j^\star} < 1 - \epsilon$$

we then obtain a contradiction with (4.18), because $\widetilde{\mathsf{P}}(\xi_j > 1 - \epsilon) > 0$. Thus for any $\epsilon > 0,$ we must have that $\mathsf{P}(\mathcal{C}_j^\epsilon) = 0$. This in turn implies that

$$1_{\{0\le j<\tau^\star\}}\left(\mathcal{S}_{\tau^\star} - \mathcal{S}_j\right) \ge 0 \quad \text{a.s.}$$

However, the $\mathcal{F}_j$-conditional expectation of the left-hand-side is zero (Doob's sampling theorem). Hence,

$$1_{\{0\le j<\tau^\star\}}\mathcal{S}_{\tau^\star} = 1_{\{0\le j<\tau^\star\}}\mathcal{S}_j \quad \text{a.s.}$$

by the sandwich property. Since j was arbitrary, this obviously implies that

$$\mathcal{S}_j = 0 \quad \text{for} \quad 0 \le j \le \tau^\star. \tag{4.19}$$

Let us next assume that for some $0 \le j \le J$, $\mathsf{P}(\tau^\star < j \le J) > 0$. We then have due to (3.7) and (4.17),

$$\xi_j 1_{\{\tau^\star<j\le J\}} \le \left(1 + \frac{\mathcal{S}_{\tau^\star} - \mathcal{S}_j}{Y_j^\star - Z_j + A_j^\star}\right) 1_{\{\tau^\star<j\le J\}} \quad \text{almost surely.} \tag{4.20}$$

For $\tau^\star < j \le J$, (2.16) implies that $\mathcal{S}_{\tau^\star} - \mathcal{S}_j \ge Z_j - Y_j^\star - A_j^\star$, where it is noted that $Z_j - Y_j^\star - A_j^\star < 0$ due to $Z_j \le Y_j$ and $A^\star_{\tau^\star+1} > 0$. Similarly, we next assume that for some $\epsilon > 0$ the set

$$\mathcal{D}_j^\epsilon := \{\tau^\star < j \le J\} \cap \left\{0 > -\epsilon(Y_j^\star - Z_j + A_j^\star) > \mathcal{S}_{\tau^\star} - \mathcal{S}_j \ge Z_j - Y_j^\star - A_j^\star\right\}$$

has positive probability. Then on $\mathcal{D}_j^\epsilon$ on has

$$0 \le 1 + \frac{\mathcal{S}_{\tau^\star} - \mathcal{S}_j}{Y_j^\star - Z_j + A_j^\star} < 1 - \epsilon,$$

which gives a contradiction with (4.20) however because $\widetilde{\mathsf{P}}(\xi_j > 1 - \epsilon) > 0$. We so conclude that

$$1_{\{\tau^\star < j \le J\}} \left(\mathcal{S}_{\tau^\star} - \mathcal{S}_j\right) \ge 0 \quad \text{a.s.}$$

and by taking the $\mathcal{F}_j$-conditional expectation again, that $\mathcal{S}_{\tau^\star} = \mathcal{S}_j$ for $\tau^\star \le j \le J$. We had already (4.19), and therefore we finally conclude that $\mathcal{S} = 0$, hence $M = M^\star$.

4.9 Proof of Corollary 3.3

If $M = M^\star$ one has $\mathrm{Var}\left(\max_{0\le j\le J}(Z_j - \widetilde{M}_j)\right) = 0$ due to Proposition 3.1. Let us now take $M \in \mathcal{M}^{\circ,0}$ with $M \ne M^\star$ and assume that $\mathrm{Var}\left(\max_{0\le j\le J}(Z_j - \widetilde{M}_j)\right) = 0$. From here we will derive a contradiction. As in the proof of Theorem 3.2 we write

$$\max_{0\le j\le J}(Z_j - \widetilde{M}_j) = Y_0^\star + \max_{0\le j\le J}(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star), \quad \text{whence}$$

$$\mathrm{Var}\left(\max_{0\le j\le J}(Z_j - \widetilde{M}_j)\right) = \mathrm{Var}\left(\max_{0\le j\le J}(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star)\right) = 0. \tag{4.21}$$

Now, $M \ne M^\star$ implies by Theorem 3.2 that

$$\widetilde{\mathsf{E}}\left[\max_{0\le j\le J}(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star)\right] > 0. \tag{4.22}$$

That is, due to (4.21) and (4.22), there exists a constant $c > 0$ such that

$$\max_{0\le j\le J}(\mathcal{S}_j + \eta_j + Z_j - Y_j^\star - A_j^\star) = c > 0.$$

Using (3.7) and the fact that always $Y_j^\star - Z_j + A_j^\star \ge 0$ and $\xi_j \le 1$, this implies

$$0 < c = \max_{0\le j\le J}(\mathcal{S}_j + (\xi_j - 1)\,(Y_j^\star - Z_j + A_j^\star)) \le \max_{0\le j\le J}(\mathcal{S}_j). \tag{4.23}$$

Consider the stopping time $\sigma := \inf\{j \ge 0 : \mathcal{S}_j \ge c\}$. Then, using $\mathcal{S}_0 = 0$ and (4.23), we must have that $0 < \sigma \le J$ almost surely. Since $\mathcal{S}$ is a martingale, Doob's sampling theorem then implies $0 = \mathcal{S}_0 = \mathsf{E}\left[\mathcal{S}_\sigma\right] \ge c$, hence a contradiction. That is, the assumption $\mathrm{Var}\left(\max_{0\le j\le J}(Z_j - \widetilde{M}_j)\right) = 0$ was false.

5 Numerical examples

5.1 Simple stylized numerical example

We first reconsider the stylized test example due to [13, Section 8], also considered in [4], where $J = 2, Z_0 = 0, Z_2 = 1$, and $Z_1 = \mathcal{U}$ is a random variable which uniformly distributed on the interval $[0, 2]$. The optimal stopping time τ^* is thus given by

$$\tau^* = \begin{cases} 1, & \mathcal{U} \geq 1, \\ 2, & \mathcal{U} < 1. \end{cases}$$

and the optimal value is $Y_0^\star = \mathsf{E}\max(\mathcal{U}, 1) = 5/4$. Furthermore, it is easy to see that the Doob martingale is given by

$$M_0^\star = 0, \quad M_1^\star = M_2^\star = \max\{\mathcal{U}, 1\} - \frac{5}{4}.$$

As an illustration of the theory developed in Sections 2-3, let us consider the linear span $M(\alpha) = \alpha M^\star$ as a pool of candidate martingales and randomize it according to (3.7). We thus consider the objective function

$$\mathcal{O}_\theta(\alpha) := \widetilde{\mathsf{E}}\Big[\max_{0\leq j\leq 2}\left(Z_j - \alpha M_j^\star + \theta\xi_j\left(Y_j^\star - Z_j + A_j^\star\right)\right)\Big], \tag{5.1}$$

for some fixed $\theta \geq 0$, where (ξ_j) are i.i.d. random variables with uniform distribution on $[-1, 1]$. Note that for this example $Y_1^\star = \max(\mathcal{U}, 1), Y_2^\star = 1$, and $A_0^\star = A_1^\star = 0, A_2^\star = \max\{\mathcal{U}, 1\} - 1$, is the non-decreasing predictable process from the Doob decomposition. Moreover, it is possible to compute (5.1) in closed form (though we omit detailed expressions which can be conveniently obtained by Mathematica for instance). In Figure 1 (left panel) we have plotted (5.1) for $\theta = 0$ and $\theta = 1$, together with the objective function

$$\overline{\mathcal{O}}_1(\alpha) := \widetilde{\mathsf{E}}\Big[\max_{0\leq j\leq 2}\left(Z_j - \alpha M_j^\star + \xi_j\right)\Big],$$

due to a "naive" randomization, not based on knowledge of the factor $Y_j^\star - Z_j + A_j^\star$. Also, in Figure 1 (right panel), the relative standard deviations $\sqrt{\mathrm{Var}(\cdot)}/Y_0^\star$ of the corresponding random variables

$$\mathcal{Z}_\theta(\alpha) := \max_{0\leq j\leq 2}\left(Z_j - \alpha M_j^\star + \theta\xi_j\left(Y_j^\star - Z_j + A_j^\star\right)\right), \quad \theta = 0, 1, \text{ and}$$
$$\overline{\mathcal{Z}}_1(\alpha) := \max_{0\leq j\leq 2}\left(Z_j - \alpha M_j^\star + \xi_j\right)$$

are depicted as a function of α.

From [13, Section 8] we know that, and from the plot of $\mathcal{O}_0(\alpha)$ in Figure 1 (left panel) we see that, $M(\alpha) \in \mathcal{M}_0^\circ$ for $\alpha \in [-4, 8/3]$. On the other hand, the right panel plot shows that $\mathrm{Var}(\mathcal{Z}_0(\alpha))$ may be relatively large for $\alpha \neq 1$, and that the Doob martingale (i.e. $\alpha = 1$) is the only surely optimal one in our parametric family. Moreover, the objective function due to the optimal randomization attains its unique minimum at the Doob martingale, i.e. for $\alpha = 1$. Further, the variance of the corresponding optimally randomized estimator attains its unique minimum zero also at $\alpha = 1$. Let us note that these observations are anticipated by Theorem 3.2 and Corollary 3.3. The catch is that for each

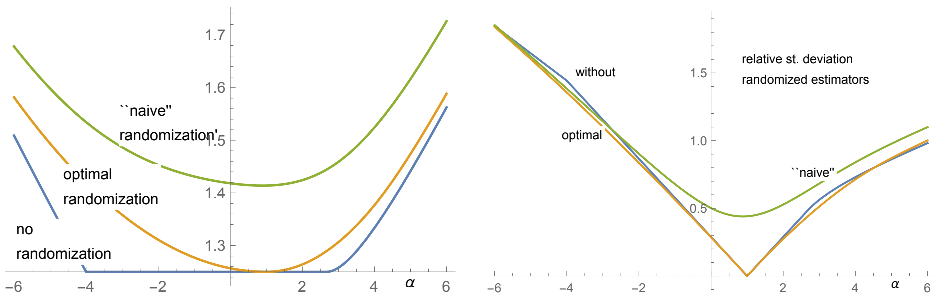

Figure 1: Left panel: objective functions $\mathcal{O}_0(\alpha)$ (no randomization), $\mathcal{O}_1(\alpha)$ (optimal randomization), and $\overline{\mathcal{O}}_1$ ("naive" randomization); right panel: relative deviations of $\mathcal{Z}_0(\alpha)$ (without randomization), $\mathcal{Z}_1(\alpha)$ (optimal randomization), $\overline{\mathcal{Z}}_1(\alpha)$ ("naive" randomization)

$\alpha \neq 1$ the randomized $M(\alpha)$ fails to be optimal in the sense of (3.6). We also see that both the optimal and the "naive" randomization render the minimization problem to be strictly convex. Moreover, while the minimum due to the "naive" randomization lays significantly above the true solution, the argument where the minimum is attained, $\overline{\alpha}$ say, identifies nonetheless a martingale that virtually coincides with the Doob optimal one. That is, $\overline{\alpha} \approx 1$ and $M(\overline{\alpha})$ is optimal corresponding to variance $\mathrm{Var}(\mathcal{Z}_0(\overline{\alpha})) \approx 0$, which can be seen in the right panel.

5.2 Bermudan call in a Black-Scholes model

In order to exhibit the merits of randomization based on the theoretical results in this paper in a more realistic case, we have constructed an example that contains all typical features of a real life Bermudan option, but, is simple enough to be treated numerically in all respects on the other hand.

As in the previous example we take $J = 2$, and specify the (discounted) cash-flows Z_j as functions of the (discounted) stock prices S_j by

$$Z_0 = 0, \quad Z_1 = (S_1 - \kappa_1)^+, \quad Z_2 = (S_2 - \kappa_2)^+ \tag{5.2}$$

For S we take the Black-Scholes model

$$S_j = S_0 \exp(-\frac{1}{2}\sigma^2 j + \sigma W_j), \quad j = 0, 1, 2, \tag{5.3}$$

where $W_1 \sim \mathcal{N}(0,1)$ and $W_{1,2} := W_2 - W_1 \sim \mathcal{N}(0,1)$, independent of W_1. As such we have a stylized example of a Bermudan call option under a Black-Scholes model with two (non-trivial) exercise dates if $\kappa_2 > \kappa_1 \geq 0$. Note that usually a Bermudan call is considered for a fixed strike and a dividend paying stock, yielding a non-trivial optimal stopping time. Though increasing strikes here look somewhat unusual, it is simple for presentation while, mathematically, the effect is the same as for a dividend paying stock and a fixed strike. For the continuation function at $j = 1$ we thus have

$$\begin{aligned} C_1(W_1) &= \mathsf{E}_{W_1}\left[\left(S_0 \exp(-\sigma^2 + \sigma W_2) - \kappa_2\right)^+\right] \\ &= \int \left(S_0 \exp(-\sigma^2 + \sigma W_1 + \sigma z) - \kappa_2\right)^+ \phi(z) dz, \end{aligned} \tag{5.4}$$

where $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ is the standard normal density. While abusing notation a bit we will denote the cash-flows by $Z_1(W_1)$ and $Z_2(W_2) = Z_2(W_1, W_{1,2})$, respectively. For the (discounted) option value at $j=0$ one thus has

$$Y_0^\star = \mathsf{E}\left[\max\left(Z_1(W_1), C_1(W_1)\right)\right] \\ = \int \max\left(\left(S_0\exp(-\frac{1}{2}\sigma^2+\sigma z)-\kappa_1\right)^+, C_1(z)\right)\phi(z)dz$$

Further we obviously have

$$Y_1^\star(W_1) = \max\left(Z_1(W_1), C_1(W_1)\right) \quad \text{and} \quad Y_2^\star(W_2) = Z_2(W_2) = Z_2(W_1,W_{1,2}).$$

The Doob martingale for this example is thus given by

$$M_0^\star = 0, \quad M_1^\star = Y_1^\star(W_1) - Y_0^\star, \quad M_2^\star - M_1^\star = Z_2(W_1,W_{1,2}) - C_1(W_1)$$

and the non-decreasing predictable component $A^\star$ is given by

$$A_0^\star = A_1^\star = 0, \quad A_2^\star = Y_1^\star(W_1) - C_1(W_1).$$

For demonstration purposes we will quasi analytically compute the optimal randomization coefficient in (3.7),

$$Y^\star - Z + A^\star = \begin{cases} Y_0^\star & j=0, \\ (C_1(W_1)-Z_1(W_1))^+, & j=1, \\ (Z_1(W_1)-C_1(W_1))^+, & j=2. \end{cases}$$

by using a Black(-Scholes) type formula

$$C_1(W_1) = S_0\exp(-\frac{1}{2}\sigma^2+\sigma W_1)\mathcal{N}\left(W_1+\frac{1}{\sigma}\ln(S_0/\kappa_2)\right) \\ - \kappa_2\mathcal{N}\left(W_1+\frac{1}{\sigma}\ln(S_0/\kappa_2)-\sigma\right),$$

and a numerical integration for obtaining the target value $Y_0^\star$. We now consider two martingale families.

(M-Sty) For any $\boldsymbol{\alpha} = (\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22})$ we set

$$M_1^{\mathsf{sty}}(\boldsymbol{\alpha},W) := \alpha_{11}\left(Y_1^\star(W_1)-Y_0^\star-W_1\right)+\alpha_{12}W_1 \\ M_2^{\mathsf{sty}}(\boldsymbol{\alpha},W) := M_1^{\mathsf{sty}}(\boldsymbol{\alpha},W)+\alpha_{21}\left(Z_2(W_1,W_{1,2})-C_1(W_1)-W_{1,2}\right)+\alpha_{22}W_{1,2}. \tag{5.5}$$

Note that $M^{\mathsf{sty}}((1,1,1,1),W) = M^\star(W)$.

(M-Hermite) Using that the (probabilistic) Hermite polynomials given by

$$He_k(x) = (-1)^k e^{\frac{x^2}{2}}\left(\frac{d}{dx}\right)^k e^{-\frac{x^2}{2}}, \quad k=0,1,2,...,$$

are orthogonal with respect to the standard Gaussian density we consider a martingale family

$$M_1^{\mathsf{H}}(\boldsymbol{\alpha},W) = \sum_{k=1}^K \alpha_{1,k}He_k(W_1) \\ M_2^{\mathsf{H}}(\boldsymbol{\alpha},W) = M_1^H(\boldsymbol{\alpha},W) + \sum_{k=0}^K\sum_{l=1}^L \alpha_{2,k,l}He_k(W_1)He_l(W_{1,2}), \tag{5.6}$$

with obvious definition of $\boldsymbol{\alpha}\in\mathbb{R}^K\oplus\mathbb{R}^{(K+1)}\times\mathbb{R}^L$ (note that $He_0\equiv 1$). Since our mere goal is to exhibit the effect of randomization, for the examples below we restrict ourselves to the choice $K=L=3$.

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The parameters in (5.2) and (5.3) are taken to be such that with a medial probability optimal exercise takes place at $j=1$. In particular, we consider two cases specified with parameter sets

$$\text{(Pa1):}\ S_0=2,\quad \sigma^2=\frac{1}{3},\quad \kappa_1=2,\quad \kappa_2=3,\quad \text{target value } Y_0^\star=0.164402,$$
$$\text{(Pa2):}\ S_0=2,\quad \sigma^2=\frac{1}{25},\quad \kappa_1=2,\quad \kappa_2=\frac{5}{2},\quad \text{target value } Y_0^\star=0.496182,$$

respectively. From Figure 2 we see that the probability of optimal exercise at $j=1$ is almost 50% for (Pa1) and almost 30% for (Pa2). Let us visualize on the basis of martingale family (M-Sty) and parameters (Pa1) the effects of randomization. Consider the objective function

$$\mathcal{O}_\theta(\boldsymbol{\alpha}) := \widetilde{\mathsf{E}}\Big[\max_{0\le j\le 2}\left(Z_j - M_j^{\mathsf{sty}}(\boldsymbol{\alpha}) + \theta\xi_j\left(Y_j^\star - Z_j + A_j^\star\right)\right)\Big]. \tag{5.7}$$

where θ scales the randomization due to i.i.d. random variables (ξ_j), uniformly distributed on $[-1,1]$. I.e., for $\theta=0$ there is no randomization and $\theta=1$ gives the optimal randomization. Now restrict (5.7) to the sub domain $\boldsymbol{\alpha}=(\alpha_1,\alpha_1,\alpha_2,\alpha_2)=:(\alpha_1,\alpha_2)$ (while slightly abusing notation), i.e. $\alpha_{11}=\alpha_{12}=\alpha_1$ and $\alpha_{21}=\alpha_{22}=\alpha_2$. The function $\mathcal{O}_0(\alpha_1,\alpha_2)$, i.e. (5.7) without randomization is visualized in Figure 3, where expectations are computed quasi-analytically with Mathematica. From this plot we see that the true value $Y_0^\star=0.164402$ is attained on the line $(\alpha_1,1)$ for various α_1 (i.e. not only in $(1,1)$). On the other hand, $\mathcal{O}_1(\alpha_1,\alpha_2)$ i.e. (5.7) with optimal randomization, has a clear strict global minimum in $(1,1)$, see Figure 4. Let us have a closer look at the map $\alpha_1\to\mathcal{O}_\theta(\alpha_1,\alpha_1,1,1)$ for $\theta=0$ and $\theta=1$, respectively, and also at $\alpha_1\to\overline{\mathcal{O}}_{0.16}(\alpha_1,\alpha_1,1,1)$ due to the "naive" randomization

$$\overline{\mathcal{O}}_{0.16}(\alpha_1,1) := \widetilde{\mathsf{E}}\Big[\max_{0\le j\le 2}\left(Z_j - M_j^{\mathsf{sty}}(\alpha_1,1)+0.16\,\xi_j\right)\Big],$$

where the scale parameter $\theta=0.16$ is taken to be roughly the option value. (It turns out that the choice of this scale factor is not critical for the location of the minimum.) In fact, the results, plotted in Figure 5, tell there own tale. The second panel depicts the relative deviation of

$$\mathcal{Z}_0(\alpha_1,1) := \max_{0\le j\le 2}\left(Z_j - M_j^{\mathsf{sty}}(\alpha_1,1)\right).$$

In fact, similar comments as for the example in Section 5.1 apply. The "naive" randomization attains its minimum at $\overline{\alpha}_1=0.9$, which we red off from the tables that generated this figure. We thus have found the martingale $M^{\mathsf{sty}}(0.9,1)$, which may be virtually considered surely optimal, as can be seen from the variance plot (second panel). Analogue visualizations for the parameter set (Pa2) with analogue conclusions may be given, though are omitted due to space restrictions.

Let us now pass on to a Monte Carlo setting, where we mimic the approach in real practice more closely. Based on N simulated samples of the underlying asset model, i.e. $S^{(n)}$, $n=1,...,N$, we consider the minimization

$$\widehat{\boldsymbol{\alpha}}_\theta := \arg\min_{\boldsymbol{\alpha}}\frac{1}{N}\sum_{n=1}^{N}\Big[\max_{0\le j\le 2}\left(Z_j^{(n)} - M_j^{(n)}(\boldsymbol{\alpha}) + \theta\xi_j\left(Y_j^{\star(n)} - Z_j^{(n)} + A_j^{\star(n)}\right)\right)\Big] \tag{5.8}$$

for $\theta=0$ (no randomization) and $\theta=1$ (optimal randomization), along with the minimization

$$\widehat{\boldsymbol{\alpha}}_{\theta^{\mathsf{naive}}} := \arg\min_{\boldsymbol{\alpha}}\frac{1}{N}\sum_{n=1}^{N}\Big[\max_{0\le j\le 2}\left(Z_j^{(n)} - M_j^{(n)}(\boldsymbol{\alpha}) + \theta_j^{\mathsf{naive}}\xi_j\right)\Big] \tag{5.9}$$

based on a "naive"randomization where the coefficients $\theta_j^{\text{naive}}, j = 0, 1, 2$ are pragmatically chosen. In (5.8) and (5.9) M stands for a generic linearly structured martingale family, such as (5.5) and (5.6) for example. The minimization problems (5.8) and (5.9) may be solved by linear programming (LP). They may be transformed into a suitable form such that the (free) LP package in R can be applied. This transformation procedure is straightforward and spelled out in [7] for example. In the latter paper it is argued that the required computation time scales with N due to the sparse structure of the coefficient matrix involved in the LP setup. However, taking advantage of this sparsity requires a special treatment of the implementation of the linear program in connection with more advanced LP solvers (as done in [7]). Since this paper is essentially on the theoretical justification of the randomized duality problem (along with the classification of optimal martingales), we consider an in-depth numerical analysis beyond scope of this paper.

For both parameter sets (Pa1) and (Pa2), and both martingale families (5.5) and (5.6) with $K = L = 3$, we have carried out the LP optimization algorithm sketched above. We have taken $N = 2000$ and for the "naive" randomization

$$\theta_0^{\text{naive}} = 1.6 \text{ for (Pa1)}, \;\; \theta_0^{\text{naive}} = 4.8 \text{ for (Pa2), and simply } \theta_1^{\text{naive}} = \theta_2^{\text{naive}} = 0.$$

In the Table 1, for (Pa1), and Table 2, for (Pa2), we present for the minimizers $\widehat{\boldsymbol{\alpha}}_0, \widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\alpha}}_{\theta^{\text{naive}}}$ the in-sample expectation $\widehat{m}$, the in-sample standard deviation $\widehat{\sigma}/\sqrt{N}$, and the path-wise maximum due to a single trajectory $\widehat{\sigma}$, followed by the corresponding "true" values m^{test}, $\sigma^{\text{test}}/\sqrt{N^{\text{test}}}$, σ^{test}, based on a large "test" simulation of $N^{\text{test}} = 10^6$ samples.

(Pa1)	M^{sty}			M^{H}		
	$\widehat{\boldsymbol{\alpha}}_0$	$\widehat{\boldsymbol{\alpha}}_{\theta^{\text{naive}}}$	$\widehat{\boldsymbol{\alpha}}_1$	$\widehat{\boldsymbol{\alpha}}_0$	$\widehat{\boldsymbol{\alpha}}_{\theta^{\text{naive}}}$	$\widehat{\boldsymbol{\alpha}}_1$
$\widehat{m}$	0.16243	0.16399	0.16403	0.16268	0.16560	0.16696
$\widehat{\sigma}/\sqrt{N}$	0.00573	0.00036	0.00029	0.00574	0.00113	0.00118
$\widehat{\sigma}$	0.25639	0.01608	0.01278	0.25676	0.05063	0.05293
m^{test}	0.16490	0.16445	0.16442	0.16709	0.16664	0.16685
$\sigma^{\text{test}}/\sqrt{N^{\text{test}}}$	0.00026	0.00001	0.00001	0.00026	0.00005	0.00005
σ^{test}	0.26096	0.01460	0.01064	0.26439	0.05083	0.05153

Table 1: LP minimization results due to M^{sty} and M^{H} for (Pa1)

(Pa2)	M^{sty}			M^{H}		
	$\widehat{\boldsymbol{\alpha}}_0$	$\widehat{\boldsymbol{\alpha}}_{\theta^{\text{naive}}}$	$\widehat{\boldsymbol{\alpha}}_1$	$\widehat{\boldsymbol{\alpha}}_0$	$\widehat{\boldsymbol{\alpha}}_{\theta^{\text{naive}}}$	$\widehat{\boldsymbol{\alpha}}_1$
$\widehat{m}$	0.48748	0.49471	0.49490	0.49329	0.50082	0.50546
$\widehat{\sigma}/\sqrt{N}$	0.02064	0.00201	0.00152	0.02076	0.00318	0.00308
$\widehat{\sigma}$	0.92301	0.08981	0.06801	0.92852	0.14222	0.13762
m^{test}	0.49820	0.49639	0.49633	0.51079	0.50870	0.50912
$\sigma^{\text{test}}/\sqrt{N^{\text{test}}}$	0.00095	0.00009	0.00007	0.00097	0.00016	0.00015
σ^{test}	0.95415	0.09038	0.06674	0.97272	0.16047	0.15103

Table 2: LP minimization results due to M^{sty} and M^{H} for (Pa2)

The results in tables Tables 1-2 show that even a simple (naive) randomization at $j = 0$ leads to a substantial variance reduction (up to 10 times) not only on training samples but also on the test ones. We think that for more structured examples and more complex families of martingales even more pronounced variance reduction effect may be expected. For example, in general it might be better to take Wiener integrals, i.e. objects of the form $\int \alpha(t, X_t)dW$, where α runs through some linear space of basis functions, as building blocks for the martingale family. Also other types of randomization can be used, for example one may take different distributions for the r.v. ξ. However all these issues will be analyzed in a subsequent study.

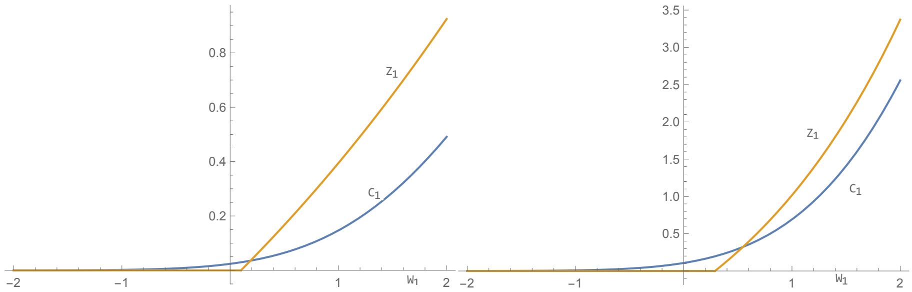

Figure 2: Cash-flow Z_1 versus continuation value C_1 as a function of W_1 for (Pa1) (left) and (Pa2) (right)

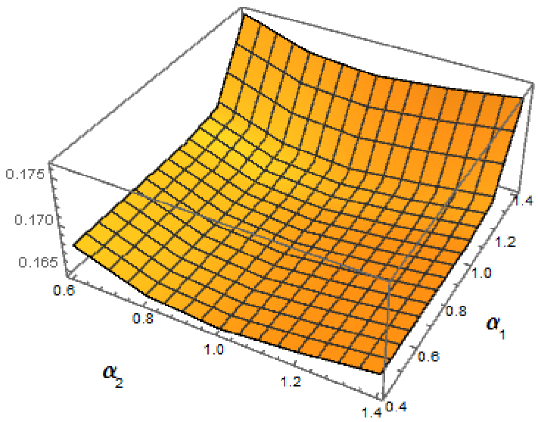

Figure 3: Object function for BS-Call (Pa1) without randomization as function of (α_1, α_2)

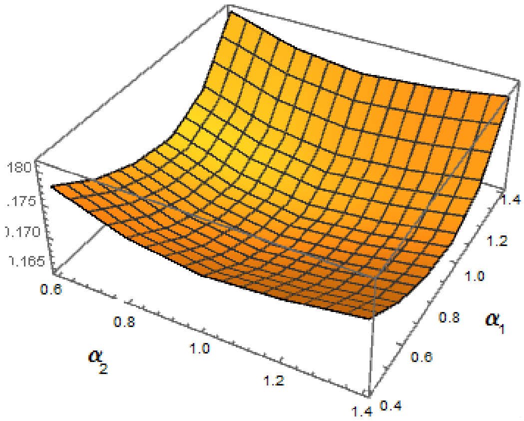

Figure 4: Object function for BS-Call (Pa1) with optimal randomization as function of (α_1, α_2)

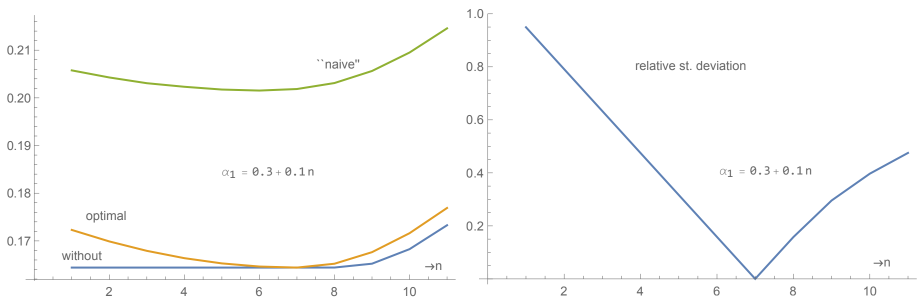

Figure 5: Left panel: object functions of α_1, with $\alpha_2 = 1$ fixed, for BS-Call (Pa1) without, optimal, and "naive" randomization; right panel: relative deviation of $\mathcal{Z}_0(\alpha_1, 1)$ (i.e. without randomization)

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