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Inductive Freeness of Ziegler's Canonical
Multiderivations for Reflection Arrangements

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INDUCTIVE FREENESS OF ZIEGLER'S CANONICAL MULTIDERIVATIONS FOR REFLECTION ARRANGEMENTS

TORSTEN HOGE AND GERHARD RÖHRLE

ABSTRACT. Let \mathcal{A} be a free hyperplane arrangement. In 1989, Ziegler showed that the restriction \mathcal{A}'' of \mathcal{A} to any hyperplane endowed with the natural multiplicity is then a free multiarrangement. We initiate a study of the stronger freeness property of inductive freeness for these canonical free multiarrangements and investigate them for the underlying class of reflection arrangements.

More precisely, let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of a complex reflection group W . By work of Terao, each such reflection arrangement is free. Thus so is Ziegler's canonical multiplicity on the restriction \mathcal{A}'' of \mathcal{A} to a hyperplane. We show that the latter is inductively free as a multiarrangement if and only if \mathcal{A}'' itself is inductively free.

CONTENTS

1. Introduction	1
2. Recollections and Preliminaries	4
3. Filtrations of Free Multiplicities	11
4. Proofs of Theorems 1.4 and 1.9	14
References	22

1. INTRODUCTION

The class of free arrangements, respectively free multiarrangements, plays a fundamental role in the theory of hyperplane arrangements, respectively multiarrangements. In his seminal work [Z89], Ziegler introduced the notion of multiarrangements and initiated the study of their freeness. We begin by recalling Ziegler's fundamental construction from *loc. cit.*

Definition 1.1. Let \mathcal{A} be an arrangement. Fix $H_0 \in \mathcal{A}$ and consider the restriction \mathcal{A}'' with respect to H_0 . Define the *canonical multiplicity* κ on \mathcal{A}'' as follows. For $Y \in \mathcal{A}''$ set

$$\kappa(Y) := |\mathcal{A}_Y| - 1,$$

i.e., $\kappa(Y)$ is the number of hyperplanes in $\mathcal{A} \setminus \{H_0\}$ lying above Y . Ziegler showed that freeness of \mathcal{A} implies freeness of (\mathcal{A}'', κ) as follows.

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Theorem 1.2 ([Z89, Thm. 11]). *Let \mathcal{A} be a free arrangement with exponents $\exp \mathcal{A} = \{1, e_2, \dots, e_\ell\}$. Let $H_0 \in \mathcal{A}$ and consider the restriction \mathcal{A}'' with respect to H_0 . Then the multiarrangement (\mathcal{A}'', κ) is free with exponents $\exp(\mathcal{A}'', \kappa) = \{e_2, \dots, e_\ell\}$.*

Note that the converse of Theorem 1.2 is false. For let \mathcal{A} be a non-free 3-arrangement, cf. [OT92, Ex. 4.34]. Since \mathcal{A}'' is of rank 2, (\mathcal{A}'', κ) is free, [Z89, Cor. 7]. Nevertheless, Ziegler's construction and in particular the question of a converse of Theorem 1.2 under suitable additional hypotheses play an important role in the study of free simple arrangements, [Y04, Thms. 2.1, 2.2], [Y05], [AY13, Cor. 4.2], [S12, Thm. 2] and [Y14, Cor. 1.35].

Because of the relevance of Ziegler's multiplicity in the theory of free arrangements, it is natural to investigate stronger freeness properties for (\mathcal{A}'', κ) and specifically to ask for an analogue of Theorem 1.2 for inductive freeness.

Question 1.3. *Is it the case that (\mathcal{A}'', κ) is inductively free whenever \mathcal{A} is?*

In this paper we initialize the study of this question and examine the multiarrangements (\mathcal{A}'', κ) where the underlying class are reflection arrangements.

It can be quite cumbersome to decide whether a given (multi)arrangement is inductively free, see for instance [BC12, §5.2], [AHR14, Lem. 4.2], [HR15, Lem. 3.5], and [CR16, Thm. 1.1]. This might entail an extensive perusal of a large number of chains of free subarrangements.

If $\mathcal{A} = \mathcal{A}(W)$ is a reflection arrangement of the complex reflection group W , then \mathcal{A} is free, thanks to work of Terao [T80b]. Thus (\mathcal{A}'', κ) is also free, by Theorem 1.2.

In our main result, we classify all instances when Ziegler's canonical multiplicity (\mathcal{A}'', κ) is inductively free in case the underlying arrangement \mathcal{A} is a reflection arrangement. In particular, we answer Question 1.3 in the affirmative for this class of arrangements.

Because of the compatibility of the product constructions for inductive freeness for simple arrangements, [HR15, Prop. 2.10], as well as for multiarrangements, [HRS17, Thm. 4] (cf. Theorem 2.23), the question of inductive freeness of (\mathcal{A}'', κ) readily reduces to the case when \mathcal{A} is irreducible. Thus for a reflection arrangement $\mathcal{A}(W)$, we may assume that W is irreducible.

Theorem 1.4. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the irreducible complex reflection group W . Let \mathcal{A}'' be the restriction of \mathcal{A} to a hyperplane in \mathcal{A} . Then (\mathcal{A}'', κ) is inductively free if and only if one of the following holds:*

- (i) \mathcal{A} is inductively free; or
- (ii) \mathcal{A} is non-inductively free of rank at most 4.

We record several consequences of Theorem 1.4. Our first observation follows from the theorem along with the classification of all inductively free restrictions of reflection arrangements from [AHR14, Thm. 1.2], see Theorem 2.4 below.

Corollary 1.5. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the complex reflection group W . Then (\mathcal{A}'', κ) is inductively free if and only if \mathcal{A}'' itself is inductively free.*

We mention that Corollary 1.5 is false in general. For the failure of the forward implication, see [HR15, Ex. 2.16]. There we present an example of an inductively free arrangement \mathcal{A} where a particular restriction to a hyperplane \mathcal{A}'' fails to be free. In particular, \mathcal{A}'' is not inductively free. Nevertheless, since \mathcal{A} is free, so is (\mathcal{A}'', κ) , thanks to Theorem 1.2. One can check that in this instance (\mathcal{A}'', κ) is actually inductively free. Nevertheless, this example is consistent with the assertion in Question 1.3. Example 2.24 shows that the reverse implication in Corollary 1.5 also fails in general. In view of these elementary examples, the equivalence of Corollary 1.5 is rather striking and underlines the very special nature of reflection arrangements.

In addition to the canonical multiplicity from Definition 1.1, we study multiplicities which are concentrated at a single hyperplane. These were introduced by Abe, Terao and Wakefield, [ATW08, §5]. While in general, (\mathcal{A}, μ) need not be free for a free hyperplane arrangement \mathcal{A} and an arbitrary multiplicity μ , e.g. see [Z89, Ex. 14], for these concentrated multiplicities freeness is also inherited from the simple arrangement. It turns out that they are closely related to Ziegler's canonical multiplicity, see Propositions 2.14 and 2.15.

Definition 1.6. Let \mathcal{A} be a simple arrangement. Fix $H_0 \in \mathcal{A}$ and $m_0 \in \mathbb{Z}_{\geq 1}$ and define the *multiplicity δ concentrated at H_0* by

$$\delta(H) := \delta_{H_0, m_0}(H) := \begin{cases} m_0 & \text{if } H = H_0, \\ 1 & \text{else.} \end{cases}$$

It turns out that in this instance, both \mathcal{A} and (\mathcal{A}, δ) inherit freeness from each other, as opposed to the general case indicated above. We record this in our next result.

Theorem 1.7. *Let \mathcal{A} be an arrangement. Fix $H_0 \in \mathcal{A}$, $m_0 \in \mathbb{Z}_{\geq 1}$ and let $\delta = \delta_{H_0, m_0}$ be the multiplicity concentrated at H_0 , as in Definition 1.6. Then \mathcal{A} is free with exponents $\exp \mathcal{A} = \{1, e_2, \dots, e_\ell\}$ if and only if (\mathcal{A}, δ) is free with exponents $\exp(\mathcal{A}, \delta) = \{m_0, e_2, \dots, e_\ell\}$.*

The forward implication of Theorem 1.7 is [ATW08, Prop. 5.2] and the reverse implication is Proposition 2.15(i).

As in the case of Ziegler's canonical multiplicity, one might also speculate whether the analogue for inductive freeness holds for the forward implication of Theorem 1.7.

Question 1.8. *Is it the case that (\mathcal{A}, δ) is inductively free whenever \mathcal{A} is?*

It turns out that Questions 1.3 and 1.8 are closely related: the assertion of Question 1.8 follows from that in Question 1.3, see Corollary 2.20.

Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the complex reflection group W . Since \mathcal{A} is free, it follows from Proposition 2.14(i) that the multiarrangement (\mathcal{A}, δ) is also free. In [HR15, Thm. 1.1], we classified all inductively free reflection arrangements. In our second main result we extend this classification to multiarrangements of the form $(\mathcal{A}(W), \delta)$. This in particular gives an affirmative answer to Question 1.8 for the class of reflection arrangements.

Theorem 1.9. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the complex reflection group W . Then (\mathcal{A}, δ) is inductively free if and only if \mathcal{A} is inductively free.*

The paper is organized as follows. In Section 2.2 we recall the fundamental results for free arrangements, in particular Terao's Addition Deletion Theorem 2.1 and the subsequent notion of an inductively free arrangement. The classification of the inductively free reflection arrangements from [HR15, Thm. 1.1], and the classification of the inductively free restrictions of reflection arrangements from [AHR14, Thm. 1.2] are restated in Section 2.3. Section 2.5 is devoted to multiarrangements and their freeness. Here we present the Addition Deletion theorem due to Abe, Terao, and Wakefield, [ATW08, Thm. 0.8]. In Section 2.6 we treat Ziegler's canonical and concentrated multiplicities. This is followed by a discussion of inductive freeness for multiarrangements in Section 2.7. Here we recall results from [HRS17] which show the compatibility of this notion with products and localization for multiarrangements that are used in the sequel.

Section 3 is central to our general arguments. Here we prove some general results for free arrangements which show that under weak conditions on the exponents of a free arrangement \mathcal{A} and a given free restriction \mathcal{A}^H , every multiplicity between the simple and the Ziegler multiplicity on \mathcal{A}^H must already be free, see Corollary 3.9. This is then applied to reflection arrangements in Proposition 3.10. We then apply these methods to the imprimitive reflection groups $G(r, 1, \ell - 1)$. Combined with the fact that the reflection arrangement $\mathcal{A}(G(r, 1, \ell - 1))$ itself is inductively free (cf. Theorem 2.3), we show that $(\mathcal{A}(G(r, 1, \ell))'', \kappa)$ is inductively free in Proposition 3.11. As a further relevant consequence for our classification, we derive that $(\mathcal{A}(W)'', \kappa)$ is inductively free, for W of type F_4 , H_4 , G_{31} and G_{32} in Corollary 3.14.

In Section 4 we present our proofs of Theorems 1.4 and 1.9. The fact that inductively free multiarrangements are closed under localization proves extremely useful in this context, see Theorem 2.22. By this means, we can for instance deduce that $(\mathcal{A}(E_6)'', \kappa)$ and $(\mathcal{A}(E_7)'', \kappa)$ are inductively free from the fact that $(\mathcal{A}(E_8)'', \kappa)$ is so, see Lemma 4.5. The same theorem in turn allows us to derive that $(\mathcal{A}(G_{34})'', \kappa)$ is not inductively free from the fact that $(\mathcal{A}(G(3, 3, 5))'', \kappa)$ isn't either, see Lemma 4.10.

Due to its large rank, the proof of inductive freeness for $(\mathcal{A}(E_8)'', \kappa)$ is the most challenging among all groups of exceptional type. It involves delicate arguments of induction, see Lemma 4.5. A crucial ingredient in our proof is the fact that on $\mathcal{A}(E_8)''$ every multiplicity between the simple one and Ziegler's multiplicity κ is free, thus readily providing free filtrations of $(\mathcal{A}(E_8)'', \kappa)$ for our analysis, see Proposition 3.10.

2. RECOLLECTIONS AND PRELIMINARIES

2.1. Hyperplane Arrangements. Let $V = \mathbb{K}^\ell$ be an ℓ -dimensional \mathbb{K} -vector space. A *hyperplane arrangement* is a pair (\mathcal{A}, V) , where \mathcal{A} is a finite collection of hyperplanes in V . Usually, we simply write \mathcal{A} in place of (\mathcal{A}, V) . We write $|\mathcal{A}|$ for the number of hyperplanes in \mathcal{A} . The empty arrangement in V is denoted by Φ_ℓ .

The *lattice* $L(\mathcal{A})$ of \mathcal{A} is the set of subspaces of V of the form $H_1 \cap \dots \cap H_i$ where $\{H_1, \dots, H_i\}$ is a subset of \mathcal{A} . For $X \in L(\mathcal{A})$, we have two associated arrangements, firstly $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$, the *localization of \mathcal{A} at X* , and secondly, the *restriction of \mathcal{A} to X* , (\mathcal{A}^X, X) , where $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$. Note that V belongs to $L(\mathcal{A})$ as the

intersection of the empty collection of hyperplanes and $\mathcal{A}^V = \mathcal{A}$. The lattice $L(\mathcal{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathcal{A})$.

If $0 \in H$ for each H in \mathcal{A} , then \mathcal{A} is called *central*. If \mathcal{A} is central, then the *center* $T_{\mathcal{A}} := \bigcap_{H \in \mathcal{A}} H$ of \mathcal{A} is the unique maximal element in $L(\mathcal{A})$ with respect to the partial order. We have a *rank* function on $L(\mathcal{A})$: $r(X) := \text{codim}_V(X)$. The *rank* $r := r(\mathcal{A})$ of \mathcal{A} is the rank of a maximal element in $L(\mathcal{A})$. The ℓ -arrangement \mathcal{A} is *essential* provided $r(\mathcal{A}) = \ell$. If \mathcal{A} is central and essential, then $T_{\mathcal{A}} = \{0\}$. Throughout, we only consider central arrangements.

2.2. Free Hyperplane Arrangements. Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V . If x_1, \dots, x_{ℓ} is a basis of V^* , then we identify S with the polynomial ring $\mathbb{K}[x_1, \dots, x_{\ell}]$. Letting S_p denote the \mathbb{K} -subspace of S consisting of the homogeneous polynomials of degree p (along with 0), S is naturally \mathbb{Z} -graded: $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ in case $p < 0$.

Let $\text{Der}(S)$ be the S -module of algebraic \mathbb{K} -derivations of S . Using the \mathbb{Z} -grading on S , $\text{Der}(S)$ becomes a graded S -module. For $i = 1, \dots, \ell$, let $D_i := \partial/\partial x_i$ be the usual derivation of S . Then D_1, \dots, D_{ℓ} is an S -basis of $\text{Der}(S)$. We say that $\theta \in \text{Der}(S)$ is *homogeneous of polynomial degree* p provided $\theta = \sum_{i=1}^{\ell} f_i D_i$, where f_i is either 0 or homogeneous of degree p for each $1 \leq i \leq \ell$. In this case we write $\text{pdeg } \theta = p$.

Let \mathcal{A} be an arrangement in V . Then for $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. The *defining polynomial* $Q(\mathcal{A})$ of \mathcal{A} is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

The *module of \mathcal{A} -derivations* of \mathcal{A} is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for each } H \in \mathcal{A}\}.$$

We say that \mathcal{A} is *free* if the module of \mathcal{A} -derivations $D(\mathcal{A})$ is a free S -module.

With the \mathbb{Z} -grading of $\text{Der}(S)$, also $D(\mathcal{A})$ becomes a graded S -module, [OT92, Prop. 4.10]. If \mathcal{A} is a free arrangement, then the S -module $D(\mathcal{A})$ admits a basis of ℓ homogeneous derivations, say $\theta_1, \dots, \theta_{\ell}$, [OT92, Prop. 4.18]. While the θ_i 's are not unique, their polynomial degrees $\text{pdeg } \theta_i$ are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement \mathcal{A} and is denoted by $\text{exp } \mathcal{A}$.

The fundamental *Addition Deletion Theorem* due to Terao [T80a] plays a crucial role in the study of free arrangements, [OT92, Thm. 4.51].

Theorem 2.1. *Suppose $\mathcal{A} \neq \Phi_{\ell}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then any two of the following statements imply the third:*

- (i) \mathcal{A} is free with $\text{exp } \mathcal{A} = \{b_1, \dots, b_{\ell-1}, b_{\ell}\}$;
- (ii) \mathcal{A}' is free with $\text{exp } \mathcal{A}' = \{b_1, \dots, b_{\ell-1}, b_{\ell} - 1\}$;
- (iii) \mathcal{A}'' is free with $\text{exp } \mathcal{A}'' = \{b_1, \dots, b_{\ell-1}\}$.

Theorem 2.1 motivates the notion of *inductively free* arrangements, cf. [OT92, Def. 4.53]:

Definition 2.2. The class \mathcal{IF} of *inductively free* arrangements is the smallest class of arrangements subject to

- (i) $\Phi_\ell \in \mathcal{IF}$ for each $\ell \geq 0$;
- (ii) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that both \mathcal{A}' and \mathcal{A}'' belong to \mathcal{IF} , and $\exp \mathcal{A}'' \subseteq \exp \mathcal{A}'$, then \mathcal{A} also belongs to \mathcal{IF} .

2.3. Reflection Arrangements. The irreducible finite complex reflection groups were classified by Shephard and Todd, [ST54]. Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group. For $w \in W$, we write $\mathrm{Fix}(w) := \{v \in V \mid wv = v\}$ for the fixed point subspace of w . For $U \subseteq V$ a subspace, we define the *parabolic subgroup* W_U of W by $W_U := \{w \in W \mid U \subseteq \mathrm{Fix}(w)\}$.

The *reflection arrangement* $\mathcal{A} = \mathcal{A}(W)$ of W in V is the hyperplane arrangement consisting of the reflecting hyperplanes of the elements in W acting as reflections on V . By Steinberg's Theorem [Ste60, Thm. 1.5], for $X \in L(\mathcal{A})$, the parabolic subgroup W_X is itself a complex reflection group, generated by the unitary reflections in W that are contained in W_X . Thus, we identify the reflection arrangement $\mathcal{A}(W_X)$ of W_X as a subarrangement of \mathcal{A} .

Using the classification and nomenclature of Shephard and Todd [ST54], we recall the main classification results from [HR15, Thm. 1.1] and [AHR14, Thm. 1.2], respectively.

Theorem 2.3. *For a finite complex reflection group W , its reflection arrangement $\mathcal{A}(W)$ is inductively free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geq 3$, $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or G_{34} .*

Theorem 2.4. *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ and let $X \in L(\mathcal{A})$. The restricted arrangement \mathcal{A}^X is inductively free if and only if one of the following holds:*

- (i) \mathcal{A} is inductively free;
- (ii) $W = G(r, r, \ell)$ and $\mathcal{A}^X \cong \mathcal{A}_p^k(r)$, where $p = \dim X$ and $p - 2 \leq k \leq p$;
- (iii) W is one of $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or G_{34} and $X \in L(\mathcal{A}) \setminus \{V\}$ with $\dim X \leq 3$.

2.4. Multiarrangements. A *multiarrangement* is a pair (\mathcal{A}, μ) consisting of a hyperplane arrangement \mathcal{A} and a *multiplicity function* $\mu : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ associating to each hyperplane H in \mathcal{A} a non-negative integer $\mu(H)$. Alternately, the multiarrangement (\mathcal{A}, μ) can also be thought of as the multiset of hyperplanes

$$(\mathcal{A}, \mu) = \{H^{\mu(H)} \mid H \in \mathcal{A}\}.$$

The *order* of the multiarrangement (\mathcal{A}, μ) is the cardinality of the multiset (\mathcal{A}, μ) ; we write $|\mu| := |(\mathcal{A}, \mu)| = \sum_{H \in \mathcal{A}} \mu(H)$. For a multiarrangement (\mathcal{A}, μ) , the underlying arrangement \mathcal{A} is sometimes called the associated *simple* arrangement, and so (\mathcal{A}, μ) itself is simple if and only if $\mu(H) = 1$ for each $H \in \mathcal{A}$.

Let (\mathcal{A}, μ) be a multiarrangement in V and let $X \in L(\mathcal{A})$. The *localization of (\mathcal{A}, μ) at X* is defined to be (\mathcal{A}_X, μ_X) , where $\mu_X = \mu|_{\mathcal{A}_X}$.

There is a natural partial order on the set of multiplicities for a given simple arrangement, as follows.

Definition 2.5. For multiplicities μ_1 and μ_2 on an arrangement \mathcal{A} , we define $\mu_1 \leq \mu_2$ provided $\mu_1(H) \leq \mu_2(H)$ for every H in \mathcal{A} .

2.5. Freeness of multiarrangements. Following Ziegler [Z89], we extend the notion of freeness to multiarrangements as follows. The *defining polynomial* $Q(\mathcal{A}, \mu)$ of the multiarrangement (\mathcal{A}, μ) is given by

$$Q(\mathcal{A}, \mu) := \prod_{H \in \mathcal{A}} \alpha_H^{\mu(H)},$$

a polynomial of degree $|\mu|$ in S .

The *module of \mathcal{A} -derivations* of (\mathcal{A}, μ) is defined by

$$D(\mathcal{A}, \mu) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{\mu(H)} S \text{ for each } H \in \mathcal{A}\}.$$

We say that (\mathcal{A}, μ) is *free* if $D(\mathcal{A}, \mu)$ is a free S -module, [Z89, Def. 6].

As in the case of simple arrangements, $D(\mathcal{A}, \mu)$ is a \mathbb{Z} -graded S -module and thus, if (\mathcal{A}, μ) is free, there is a homogeneous basis $\theta_1, \dots, \theta_\ell$ of $D(\mathcal{A}, \mu)$. The multiset of the unique polynomial degrees $\text{pdeg } \theta_i$ forms the set of *exponents* of the free multiarrangement (\mathcal{A}, μ) and is denoted by $\text{exp}(\mathcal{A}, \mu)$.

Next we recall Ziegler's analogue of Saito's criterion.

Theorem 2.6 ([Z89, Thm. 8]). *Let $\theta_1, \dots, \theta_\ell$ be in $D(\mathcal{A}, \mu)$. Then $\{\theta_1, \dots, \theta_\ell\}$ is a basis of $D(\mathcal{A}, \mu)$ if and only if*

$$(2.7) \quad \det M(\theta_1, \dots, \theta_\ell) \doteq Q(\mathcal{A}, \mu).$$

In particular, if $\theta_1, \dots, \theta_\ell$ are all homogeneous, then $\{\theta_1, \dots, \theta_\ell\}$ is a basis of $D(\mathcal{A}, \mu)$ if and only if $\theta_1, \dots, \theta_\ell$ are independent over S and

$$(2.8) \quad \sum \text{pdeg } \theta_i = \deg Q(\mathcal{A}, \mu) = |\mu|.$$

Here the notation \doteq indicates equality up to a non-zero scalar multiple and $M(\theta_1, \dots, \theta_\ell)$ is the coefficient matrix of $\{\theta_1, \dots, \theta_\ell\}$, see [OT92, Def. 4.11].

Note that, owing to (2.7), if (\mathcal{A}, μ) is free, one can recover μ from the basis $\{\theta_1, \dots, \theta_\ell\}$.

We recall the construction from [ATW08] for the counterpart of Theorem 2.1 in this more general setting.

Definition 2.9. Let $(\mathcal{A}, \mu) \neq \Phi_\ell$ be a multiarrangement. Fix H_0 in \mathcal{A} . We define the *deletion* (\mathcal{A}', μ') and *restriction* (\mathcal{A}'', μ^*) of (\mathcal{A}, μ) with respect to H_0 as follows. If $\mu(H_0) = 1$, then set $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and define $\mu'(H) = \mu(H)$ for all $H \in \mathcal{A}'$. If $\mu(H_0) > 1$, then set $\mathcal{A}' = \mathcal{A}$ and define $\mu'(H_0) = \mu(H_0) - 1$ and $\mu'(H) = \mu(H)$ for all $H \neq H_0$.

Let $\mathcal{A}'' = \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. The *Euler multiplicity* μ^* of \mathcal{A}'' is defined as follows. Let $Y \in \mathcal{A}''$. Since the localization \mathcal{A}_Y is of rank 2, the multiarrangement (\mathcal{A}_Y, μ_Y) is free, [Z89, Cor. 7]. According to [ATW08, Prop. 2.1], the module of derivations $D(\mathcal{A}_Y, \mu_Y)$ admits a particular homogeneous basis $\{\theta_Y, \psi_Y, D_3, \dots, D_\ell\}$, such that $\theta_Y \notin \alpha_0 \text{Der}(S)$ and

$\psi_Y \in \alpha_0 \text{Der}(S)$, where $H_0 = \ker \alpha_0$. Then on Y the Euler multiplicity μ^* is defined to be $\mu^*(Y) = \text{pdeg } \theta_Y$.

Often, (\mathcal{A}, μ) , (\mathcal{A}', μ') and (\mathcal{A}'', μ^*) is referred to as the *triple* of (\mathcal{A}, μ) with respect to H_0 .

We require some core results from [ATW08].

Theorem 2.10 ([ATW08, Thm. 0.4]). *Suppose that $(\mathcal{A}, \mu) \neq \Phi_\ell$. Fix H_0 in \mathcal{A} . If both (\mathcal{A}, μ) and (\mathcal{A}', μ') are free, then there exists a basis $\{\theta_1, \dots, \theta_\ell\}$ of $D(\mathcal{A}', \mu')$ such that $\{\theta_1, \dots, \alpha_k \theta_k, \dots, \theta_\ell\}$ is a basis of $D(\mathcal{A}, \mu)$ for some $1 \leq k \leq \ell$.*

Fix $H_0 = \ker \alpha_0$ in \mathcal{A} and let \mathcal{A}'' be the restriction with respect to H_0 . Consider the projection $S \rightarrow \bar{S} := S/\alpha_0 S$, $f \mapsto \bar{f}$. Following [ATW08], there is a canonical map $D(\mathcal{A}, \mu) \rightarrow D(\mathcal{A}'', \mu^*)$, $\theta \mapsto \bar{\theta}$, where $\bar{\theta}(\bar{f}) := \overline{\theta(f)}$, for $\bar{f} \in \bar{S}$.

Theorem 2.11 ([ATW08, Thm. 0.6]). *Suppose that $(\mathcal{A}, \mu) \neq \Phi_\ell$. Fix H_0 in \mathcal{A} . Suppose that both (\mathcal{A}, μ) and (\mathcal{A}', μ') are free. Let $\{\theta_1, \dots, \theta_\ell\}$ be a basis of $D(\mathcal{A}', \mu')$ as in Theorem 2.10. Then $\{\bar{\theta}_1, \dots, \bar{\theta}_{k-1}, \bar{\theta}_{k+1}, \dots, \bar{\theta}_\ell\}$ is a basis of $D(\mathcal{A}'', \mu^*)$.*

Theorem 2.12 ([ATW08, Thm. 0.8] Addition Deletion Theorem for Multiarrangements). *Suppose that $(\mathcal{A}, \mu) \neq \Phi_\ell$. Fix H_0 in \mathcal{A} and let (\mathcal{A}, μ) , (\mathcal{A}', μ') and (\mathcal{A}'', μ^*) be the triple with respect to H_0 . Then any two of the following statements imply the third:*

- (i) (\mathcal{A}, μ) is free with $\exp(\mathcal{A}, \mu) = \{b_1, \dots, b_{\ell-1}, b_\ell\}$;
- (ii) (\mathcal{A}', μ') is free with $\exp(\mathcal{A}', \mu') = \{b_1, \dots, b_{\ell-1}, b_\ell - 1\}$;
- (iii) (\mathcal{A}'', μ^*) is free with $\exp(\mathcal{A}'', \mu^*) = \{b_1, \dots, b_{\ell-1}\}$.

In order to apply Theorem 2.12 effectively, it is crucial to be able to determine the Euler multiplicity for the restrictions at hand. The following is part of [ATW08, Prop. 4.1] relevant for our purposes.

Proposition 2.13. *Let $H_0 \in \mathcal{A}$, $\mathcal{A}'' = \mathcal{A}^{H_0}$ and let $X \in \mathcal{A}''$. Let μ be a multiplicity on \mathcal{A} . Let $\mu_0 = \mu(H_0)$. Further let $k = |\mathcal{A}_X|$ and $\mu_1 = \max\{\mu(H) \mid H \in \mathcal{A}_X \setminus \{H_0\}\}$.*

- (1) *If $k = 2$, then $\mu^*(X) = \mu_1$.*
- (2) *If $k = 3$, $2\mu_0 \leq |\mu_X|$, and $2\mu_1 \leq |\mu_X|$, then $\mu^*(X) = \lfloor |\mu_X|/2 \rfloor$.*
- (3) *If $|\mu_X| \leq 2k - 1$ and $\mu_0 > 0$, then $\mu^*(X) = k - 1$.*

2.6. Ziegler's Multiplicity and concentrated Multiplicities. Recall Ziegler's multiplicity κ from Definition 1.1 and concentrated multiplicities from Definition 1.6. The following combines [ATW08, Prop. 5.2], parts of its proof and Theorem 1.2. The proof of Proposition 2.14(i) given in [ATW08] depends on Theorem 2.12. In [HRS17, Prop. 1], we presented an elementary explicit construction for a homogeneous S -basis of $D(\mathcal{A}, \delta)$.

Proposition 2.14. *Let \mathcal{A} be a free arrangement with exponents $\exp \mathcal{A} = \{1, e_2, \dots, e_\ell\}$. Fix $H_0 \in \mathcal{A}$, $m_0 \in \mathbb{Z}_{\geq 1}$ and let $\delta = \delta_{H_0, m_0}$ be as in Definition 1.6. Let $(\mathcal{A}'', \delta^*)$ be the restriction with respect to H_0 . Then we have*

- (i) (\mathcal{A}, δ) is free with exponents $\exp(\mathcal{A}, \delta) = \{m_0, e_2, \dots, e_\ell\}$;
- (ii) $(\mathcal{A}'', \delta^*) = (\mathcal{A}'', \kappa)$ is free with exponents $\exp(\mathcal{A}'', \kappa) = \{e_2, \dots, e_\ell\}$.

Part (i) of the following is a converse of Proposition 2.14(i).

Proposition 2.15. *Let \mathcal{A} be an arrangement. Fix $H_0 \in \mathcal{A}$, $m_0 \in \mathbb{Z}_{\geq 1}$ and let $\delta = \delta_{H_0, m_0}$ be as in Definition 1.6. Let $(\mathcal{A}'', \delta^*)$ be the restriction with respect to H_0 . Suppose that (\mathcal{A}, δ) is free with exponents $\exp(\mathcal{A}, \delta) = \{m_0, e_2, \dots, e_\ell\}$. Then we have*

- (i) \mathcal{A} is free with exponents $\exp \mathcal{A} = \{1, e_2, \dots, e_\ell\}$.
- (ii) $(\mathcal{A}'', \delta^*) = (\mathcal{A}'', \kappa)$ is free with exponents $\exp(\mathcal{A}'', \kappa) = \{e_2, \dots, e_\ell\}$.

Proof. Clearly, part (ii) follows from part (i) and Proposition 2.14(ii).

The proof of (i) is analogous to the one of [OT92, Prop. 4.27]. Let $\alpha_0 \in V^*$ with $H_0 = \ker \alpha_0$. Now let

$$\text{Ann}(H_0) = \{\theta \in D(\mathcal{A}, \delta) \mid \theta(\alpha_0) = 0\}$$

be the annihilator of H_0 in $D(\mathcal{A}, \delta)$ which is a graded S -submodule of $D(\mathcal{A}, \delta)$. Let θ_E be the Euler derivation in $\text{Der}(S)$ [OT92, Def. 4.7]. Define

$$\theta_\delta := \alpha_0^{m_0-1} \theta_E.$$

Then θ_δ belongs to $D(\mathcal{A}, \delta)$. Moreover for each $\theta \in D(\mathcal{A}, \delta)$, we have

$$\theta - \frac{\theta(\alpha_0)}{\alpha_0^{m_0}} \theta_\delta \in \text{Ann}(H_0).$$

Therefore, since

$$S\theta_\delta \cap \text{Ann}(H_0) = \{0\},$$

we derive that

$$D(\mathcal{A}, \delta) = S\theta_\delta \oplus \text{Ann}(H_0)$$

is a direct sum of S -modules. Let $\{\theta_2, \dots, \theta_\ell\}$ be a minimal system of homogeneous generators of the S -module $\text{Ann}(H_0)$. Then $\{\theta_\delta, \theta_2, \dots, \theta_\ell\}$ is a minimal system of homogeneous generators of $D(\mathcal{A}, \delta)$. It follows from [OT92, Thm. A.19], that $\{\theta_\delta, \theta_2, \dots, \theta_\ell\}$ is a homogeneous S -basis of $D(\mathcal{A}, \delta)$.

Since $\text{Ann}(H_0) \subset D(\mathcal{A}, \delta) \subset D(\mathcal{A})$ are S -submodules, $\{\theta_2, \dots, \theta_\ell\}$ is linearly independent over S in $D(\mathcal{A})$. Since $|\delta| = |\mathcal{A}| + m_0 - 1$ and $\text{pdeg } \theta_\delta = m_0$, it follows from Ziegler's analogue of Saito's criterion (3.13) that $\sum_{i=2}^{\ell} \text{pdeg } \theta_i = |\mathcal{A}| - 1$. Clearly, θ_E belongs to $D(\mathcal{A})$ but not to $\text{Ann}(H_0)$. Consequently, it follows from Saito's criterion, [OT92, Thm. 4.23], that \mathcal{A} is free with $\{\theta_E, \theta_2, \dots, \theta_\ell\}$ a homogeneous S -basis of $D(\mathcal{A})$. \square

Note that the proof of Proposition 2.15 shows that m_0 is necessarily an exponent of (\mathcal{A}, δ) if the latter is free (so it does no harm to require this in the hypothesis).

The following is a special case of a general compatibility result of the Euler multiplicity with localizations from [HRS17, Lem. 2].

Lemma 2.16. *Let $X \in L(\mathcal{A})$, $H_0 \in \mathcal{A}_X$, and let \mathcal{A}'' be the restriction with respect to H_0 . Let $\delta = \delta_{H_0, m_0}$ be as in Definition 1.6. Then we have*

- (i) $((\mathcal{A}_X)'' , (\delta_X)^*) = ((\mathcal{A}'')_X , (\delta^*)_X)$, and
- (ii) $((\mathcal{A}_X)'' , \kappa) = ((\mathcal{A}'')_X , \kappa_X)$ (where κ on the left is the canonical multiplicity resulting from restriction of \mathcal{A}_X to H_0).

Proof. Part (i) follows from *loc. cit.* Part (ii) follows from part (i) together with the fact that $((\mathcal{A}_X)'' , (\delta_X)^*) = ((\mathcal{A}_X)'' , \kappa)$, cf. Proposition 2.14(ii). \square

2.7. Inductive Freeness for Multiarrangements. As in the simple case, Theorem 2.12 motivates the notion of inductive freeness.

Definition 2.17 ([ATW08, Def. 0.9]). The class \mathcal{IFM} of *inductively free* multiarrangements is the smallest class of arrangements subject to

- (i) $\Phi_\ell \in \mathcal{IFM}$ for each $\ell \geq 0$;
- (ii) for a multiarrangement (\mathcal{A}, μ) , if there exists a hyperplane $H_0 \in \mathcal{A}$ such that both (\mathcal{A}', μ') and (\mathcal{A}'', μ^*) belong to \mathcal{IFM} , and $\exp(\mathcal{A}'', \mu^*) \subseteq \exp(\mathcal{A}', \mu')$, then (\mathcal{A}, μ) also belongs to \mathcal{IFM} .

Remark 2.18. As for simple arrangements, if $r(\mathcal{A}) \leq 2$, then (\mathcal{A}, μ) is inductively free, [Z89, Cor. 7].

Remark 2.19. In analogy to the simple case, cf. [OT92, §4.3, p. 119], [HR15, Rem. 2.9], it is possible to describe an inductively free multiarrangement (\mathcal{A}, μ) by means of a so called *induction table*. In this process we start with an inductively free multiarrangement (frequently Φ_ℓ) and add hyperplanes successively ensuring that part (ii) of Definition 2.17 is satisfied. We refer to this process as *induction of hyperplanes*. This procedure amounts to choosing an order on consecutive multiplicities μ_i of the simple arrangement \mathcal{A} for $i = 1, \dots, n = |\mu|$ (see Definition 2.5), so that $|\mu_i| = i$, $\mu_n = \mu$, and each restriction (\mathcal{A}'', μ_i^*) is inductively free. As in the simple case, in the associated induction table we record in the i -th row the information of the i -th step of this process, by listing $\exp(\mathcal{A}', \mu'_i) = \exp(\mathcal{A}, \mu_{i-1})$, the defining form α_{H_i} of H_i , as well as $\exp(\mathcal{A}'', \mu_i^*)$. For instance, see Tables 1, 3, and 4 below.

We record an easy consequence of Proposition 2.14(ii) in the context of inductive freeness which is going to be very useful in our proof of Theorem 1.9.

Corollary 2.20. *Suppose that both \mathcal{A} and (\mathcal{A}'', κ) are inductively free. Then (\mathcal{A}, δ) is inductively free.*

Proof. The result follows readily from induction on $m_0 \geq 1$, Proposition 2.14(ii) and a repeated application of the addition part of Theorem 2.12. \square

The following special case of Corollary 2.20 is immediate from [Z89, Cor. 7].

Corollary 2.21. *Suppose that the rank 3 arrangement \mathcal{A} is inductively free. Then (\mathcal{A}, δ) is inductively free.*

The following is a very useful tool for showing that a given multiarrangement is not inductively free by exhibiting a localization which fails to be inductively free.

Theorem 2.22 ([HRS17, Thm. 3]). *The class \mathcal{IFM} is closed under taking localizations.*

We also require the fact that inductive freeness for multiarrangements behave well with the product construction.

Theorem 2.23 ([HRS17, Thm. 4]). *A product of multiarrangements belongs to \mathcal{IFM} if and only if each factor belongs to \mathcal{IFM} .*

We close this section with an example which shows that the reverse implication of Corollary 1.5 fails in general.

Example 2.24. Consider the complex 4-arrangement \mathcal{A} given by the defining polynomial $Q(\mathcal{A}) := xyzt(x-y)(x-z)(x-y+t)(x-z+t)$. Then one checks that \mathcal{A}^{H_t} is inductively free, while both \mathcal{A} and $(\mathcal{A}^{H_t}, \kappa)$ are not free.

3. FILTRATIONS OF FREE MULTIPLICITIES

In the sequel, we denote by $\mathbb{1}$ the simple multiplicity on a given arrangement. Recall the partial order on the set of multiplicities on an arrangement from Definition 2.5.

Definition 3.1. Let \mathcal{A} be a free arrangement. Suppose there is a free multiplicity $\mu > \mathbb{1}$ on \mathcal{A} such that there is a sequence of free multiplicities μ_i on \mathcal{A} satisfying $\mu_i < \mu_{i+1}$ and $|\mu_{i+1}| = |\mu_i| + 1$, for $i = 1, \dots, n-1$, where $\mu_1 := \mathbb{1}$ and $\mu_n := \mu$. Then we say that the sequence of multiarrangements (\mathcal{A}, μ_i) is a *free filtration* of (\mathcal{A}, μ) or simply that the sequence μ_i is a *filtration* of free multiplicities on \mathcal{A} .

Lemma 3.2. *Let \mathcal{A} be an inductively free arrangement. Suppose there is a free multiplicity $\mu > \mathbb{1}$ on \mathcal{A} along with a free filtration $\mathbb{1} =: \mu_1 < \dots < \mu_n := \mu$ of (\mathcal{A}, μ) as in Definition 3.1. If each restriction along the free filtration is inductively free, then so is (\mathcal{A}, μ) .*

Proof. It follows from Theorem 2.10 that the exponents of $(\mathcal{A}', \mu'_i) = (\mathcal{A}, \mu_{i-1})$ and of (\mathcal{A}, μ_i) differ in precisely one entry by 1. So that, by Theorem 2.12, the exponents of each restriction along the free filtration satisfy $\exp(\mathcal{A}'', \mu_i^*) \subseteq \exp(\mathcal{A}', \mu'_i)$. Now, since \mathcal{A} as well as each restriction (\mathcal{A}'', μ_i^*) is inductively free, a repeated application of the addition part of Theorem 2.12 gives the desired result. \square

We record a special case of Lemma 3.2.

Corollary 3.3. *Let \mathcal{A} be an inductively free 3-arrangement. Suppose there is a free multiplicity $\mu \geq \mathbb{1}$ on \mathcal{A} along with a free filtration as in Lemma 3.2. Then (\mathcal{A}, μ) is inductively free.*

Proof. Since each restriction along the free filtration is of rank 2, it is already inductively free, by Remark 2.18. So the result is immediate from Corollary 3.3. \square

In our next result we present a mild condition on a free multiplicity μ of a free arrangement \mathcal{A} which implies that every intermediate multiplicity $\mathbb{1} < \nu < \mu$ is also free.

Lemma 3.4. *Let \mathcal{A} be a free arrangement with exponents $1 \leq e_2 \leq \dots \leq e_\ell$. Assume that there is a free multiplicity $\mu > \mathbb{1}$ on \mathcal{A} with $\exp(\mathcal{A}, \mu) = \{e, e_2, \dots, e_\ell\}$, where $e = 1 + |\mu| - |\mathcal{A}|$. Suppose that $|\mu| - |\mathcal{A}| \geq e_\ell$. Let ν be a multiplicity satisfying $\mathbb{1} < \nu < \mu$. Then (\mathcal{A}, ν) is free with $\exp(\mathcal{A}, \nu) = \{\tilde{e}, e_2, \dots, e_\ell\}$, where $\tilde{e} = 1 + |\nu| - |\mathcal{A}|$.*

Moreover, if $\theta_1, \dots, \theta_\ell$ is a homogeneous basis of $D(\mathcal{A}, \mu)$ with $\text{pdeg } \theta_i = e_i$ for $i = 2, \dots, \ell$ and $\text{pdeg } \theta_1 = e$, then

$$(3.5) \quad \left(\prod_{H \in \mathcal{A}} \alpha_H^{\nu(H)-1} \right) \theta_E, \theta_2, \dots, \theta_\ell$$

is a basis of $D(\mathcal{A}, \nu)$.

Proof. Let $\theta_1, \dots, \theta_\ell$ be a homogeneous basis of $D(\mathcal{A}, \mu)$ with $\text{pdeg } \theta_i = e_i$ for $i = 2, \dots, \ell$ and $\text{pdeg } \theta_1 = e = 1 + |\mu| - |\mathcal{A}|$. It is enough to show that $\theta_E, \theta_2, \dots, \theta_\ell$ is a basis of $D(\mathcal{A})$. For, then the derivations given in (3.5) are linearly independent over S and obviously also members of $D(\mathcal{A}, \nu)$ (as $\nu \leq \mu$) so that the sum over their polynomial degrees equals $|\nu|$, as $\sum_{i=2}^{\ell} e_i = |\mathcal{A}| - 1$. We are going to check that the conditions of [OT92, Thm. 4.42] hold. For that we have to show that

$$(3.6) \quad \theta_i \notin S\theta_E + S\theta_2 + \dots + S\theta_{i-1}$$

holds for $i = 2, \dots, \ell$. Assume that (3.6) fails for some i . Then there are polynomials $p, p_2, \dots, p_{i-1} \in S$ such that

$$(3.7) \quad p\theta_E = \sum_{k=2}^{i-1} p_k\theta_k + \theta_i.$$

Consequently, since $\theta_2, \dots, \theta_i$ are linearly independent over S , $p \neq 0$. Now applying (3.7) to α_H for $H \in \mathcal{A}$, we get that $p \cdot \alpha_H \in \alpha_H^{\mu(H)} S$. Hence

$$p \in \left(\prod_{H \in \mathcal{A}} \alpha_H^{\mu(H)-1} \right) S.$$

Therefore, $\deg p \geq \sum_{H \in \mathcal{A}} (\mu(H) - 1) = |\mu| - |\mathcal{A}|$. By (3.7), we have $1 + \deg p = \text{pdeg}(p\theta_E) = \text{pdeg } \theta_i$. Using the hypothesis $|\mu| - |\mathcal{A}| \geq e_\ell$, we get

$$\text{pdeg } \theta_i \geq 1 + |\mu| - |\mathcal{A}| > |\mu| - |\mathcal{A}| \geq e_\ell = \text{pdeg } \theta_\ell \geq \text{pdeg } \theta_i,$$

which is absurd. Consequently, (3.6) holds for all $i = 2, \dots, \ell$ and so $\theta_E, \theta_2, \dots, \theta_\ell$ is a basis of $D(\mathcal{A})$, as claimed, by *loc. cit.* \square

We record an important consequence of Lemma 3.4 and Theorem 2.12 which shows that the multisets of exponents of the restrictions along a free filtration, as in Lemma 3.4, are constant and do not depend on the multiplicities μ_i^* .

Corollary 3.8. *Let \mathcal{A} and (\mathcal{A}, μ) be as in Lemma 3.4. Let $\mathbb{1} =: \mu_1 < \dots < \mu_n := \mu$ be a free filtration of (\mathcal{A}, μ) as in Definition 3.1. Then for each restriction, we have $(\mathcal{A}'', \mu_i^*) = (\mathcal{A}'', \kappa)$, where κ is Ziegler's canonical multiplicity on \mathcal{A} . In particular, each such restriction along the filtration is free with $\exp(\mathcal{A}'', \mu_i^*) = \exp(\mathcal{A}'', \kappa) = \{e_2, \dots, e_\ell\}$, where $\exp \mathcal{A} = \{1, e_2, \dots, e_\ell\}$.*

Proof. Let $H_0 \in \mathcal{A}$ and let $\mathbb{1} \leq \nu' < \nu \leq \mu$ be free multiplicities as given by Lemma 3.4, where (\mathcal{A}', ν') is the deletion of (\mathcal{A}, ν) with respect to H_0 . Let $\theta_1, \theta_2, \dots, \theta_\ell$ be a free basis of $D(\mathcal{A}', \nu')$. It follows from Lemma 3.4 that $\alpha_0 \theta_1, \theta_2, \dots, \theta_\ell$ is a free basis of $D(\mathcal{A}, \nu)$, as given by (3.5), where $\ker \alpha_0 = H_0$. Owing to Theorem 2.11, we see that $\bar{\theta}_2, \dots, \bar{\theta}_\ell$ is a basis of $D(\mathcal{A}'', \nu^*)$.

Now let δ be the concentrated multiplicity on \mathcal{A} given by $\delta(H_0) = 2$. Then $\mathbb{1} < \delta \leq \mu$. Hence, according to Lemma 3.4, (\mathcal{A}, δ) is free with $\alpha_0 \theta_E, \theta_2, \dots, \theta_\ell$ being a basis of $D(\mathcal{A}, \delta)$, by (3.5). Thanks to Proposition 2.14, $(\mathcal{A}'', \delta^*) = (\mathcal{A}'', \kappa)$.

Since $\theta_E, \theta_2, \dots, \theta_\ell$ is a basis of $D(\mathcal{A})$, it follows again from Theorem 2.11 applied to (\mathcal{A}, δ) and $(\mathcal{A}', \delta') = (\mathcal{A}, \mathbb{1})$, that $\bar{\theta}_2, \dots, \bar{\theta}_\ell$ is also a basis of $D(\mathcal{A}'', \kappa)$. Hence, thanks to (2.7), $\nu^* = \kappa$. \square

Next we apply Lemma 3.4 in the context of Ziegler's canonical multiplicity κ .

Corollary 3.9. *Let \mathcal{A} be a free arrangement with exponents $1 = e_1 \leq e_2 \leq \dots \leq e_{\ell-1} < e_\ell$ and let $H \in \mathcal{A}$ such that \mathcal{A}^H is free with $\exp(\mathcal{A}^H) = \{1, e_2, \dots, e_{\ell-1}\}$. Then (\mathcal{A}^H, μ) is free for every multiplicity $\mathbb{1} < \mu < \kappa$ with $\exp(\mathcal{A}^H, \mu) = \{1 + |\mu| - |\mathcal{A}^H|, e_2, \dots, e_{\ell-1}\}$.*

Proof. Since \mathcal{A}^H is free with $\exp(\mathcal{A}^H) = \{1, e_2, \dots, e_{\ell-1}\}$, we have $|\mathcal{A}^H| = 1 + e_2 + \dots + e_{\ell-1}$. Also note that $|\kappa| = |\mathcal{A}| - 1$. Therefore,

$$1 + |\kappa| - |\mathcal{A}^H| = |\mathcal{A}| - (1 + e_2 + \dots + e_{\ell-1}) = e_\ell.$$

Consequently, $\exp(\mathcal{A}^H, \kappa) = \{e_\ell, e_2, \dots, e_{\ell-1}\} = \{1 + |\kappa| - |\mathcal{A}^H|, e_2, \dots, e_{\ell-1}\}$, by Theorem 1.2. Thus $|\kappa| - |\mathcal{A}^H| = e_\ell - 1 \geq e_{\ell-1}$, by hypothesis. So the result follows from Lemma 3.4 applied to \mathcal{A}^H and κ . \square

Corollary 3.9 yields that under rather weak assumptions, every multiplicity between $\mathbb{1}$ and the Ziegler multiplicity κ is free. This yields an abundance of free filtrations from $\mathbb{1}$ to κ on \mathcal{A}^H . We demonstrate this in the context of reflection arrangements.

Proposition 3.10. *Let W be of type $A_\ell, B_\ell, G(r, 1, \ell), F_4, H_4, G_{31}, G_{32}, E_6, E_7$, or E_8 . Let $\mathcal{A} = \mathcal{A}(W)$. Then for any $H \in \mathcal{A}$ and every multiplicity $\mathbb{1} \leq \mu \leq \kappa$, the multiarrangement (\mathcal{A}^H, μ) is free with $\exp(\mathcal{A}^H, \mu) = \{1 + |\mu| - |\mathcal{A}^H|, e_2, \dots, e_{\ell-1}\}$.*

Proof. It readily follows from [OT92, Props. 6.73, 6.77] and the tables in [OT92, App. C] that in the given cases any restriction \mathcal{A}^H satisfies the condition of Corollary 3.9. \square

Armed with Corollaries 3.8 and 3.9, we can now prove our first main result of our classification.

Proposition 3.11. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the full monomial group $W = G(r, 1, \ell)$ for $r, \ell \geq 2$. Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Proof. Owing to Theorem 2.3 and Corollary 2.20, it is enough to show that (\mathcal{A}'', κ) is inductively free. We prove the latter by induction on ℓ . Clearly, for $\ell \leq 3$, the statement holds. Thus suppose that $\ell \geq 4$ and that the result is true for smaller ranks.

Note that irrespective of the choice of hyperplane in \mathcal{A} , we have

$$(3.12) \quad \mathcal{A}'' \cong \mathcal{A}(G(r, 1, \ell - 1)),$$

by [OS82, Prop. 2.11] (cf. [OT92, Prop. 6.82]). Owing to [OS82, Prop. 2.13] (cf. [OT92, Prop. 6.77]), we have

$$(3.13) \quad \exp \mathcal{A} = \{1, r + 1, \dots, (\ell - 1)r + 1\}.$$

Thanks to Proposition 3.10, each multiplicity μ on \mathcal{A}'' with $\mathbb{1} \leq \mu \leq \kappa$ is free. So we pick an arbitrary free filtration of \mathcal{A}'' from $\mathbb{1}$ to κ . Since $|\kappa| - |\mathcal{A}''| = (\ell - 1)r > (\ell - 2)r + 1$, which is the highest exponent of \mathcal{A}'' , by (3.12) and (3.13), the hypotheses of Lemma 3.4 are satisfied for $\mathcal{B} := \mathcal{A}''$. Consequently, Corollary 3.8 implies that for each multiplicity μ along our free chain we have $(\mathcal{B}'', \mu^*) = (\mathcal{B}'', \kappa_1)$, where κ_1 denotes the Ziegler multiplicity on \mathcal{B}'' . Since $\mathcal{B}'' \cong \mathcal{A}(G(r, 1, \ell - 2))$, by (3.12), it follows from our induction hypothesis that $(\mathcal{B}'', \kappa_1)$ is inductively free. Consequently, since the simple arrangement \mathcal{A}'' is inductively free, thanks to Theorem 2.3, it follows that (\mathcal{A}'', κ) is inductively free, by a repeated application of the addition part of Theorem 2.12. \square

Moreover, Corollaries 3.3 and 3.9, imply the following consequence of Theorem 2.4.

Corollary 3.14. *Let W be of type F_4 , H_4 , G_{31} or G_{32} . Then $(\mathcal{A}(W)'', \kappa)$ is inductively free.*

Proof. By Proposition 3.10, in each of the given cases the condition in Corollary 3.9 on the exponents of $\mathcal{A}(W)''$ is satisfied, so that every chain of multiplicities between $\mathbb{1}$ and κ on $\mathcal{A}(W)''$ is free. By Theorem 2.4, $\mathcal{A}(W)''$ is inductively free in each instance. Therefore, it follows from Corollary 3.3 that also $(\mathcal{A}(W)'', \kappa)$ is inductively free. \square

4. PROOFS OF THEOREMS 1.4 AND 1.9

Throughout, W denotes a complex reflection group. Owing to [HR15, Prop. 2.10] and Theorem 2.23, we may assume that W is irreducible. Fix $H_0 \in \mathcal{A}$ and let \mathcal{A}'' be the restriction of \mathcal{A} with respect to H_0 . Moreover, fix $m_0 \in \mathbb{Z}_{\geq 1}$ and let $\delta = \delta_{H_0, m_0}$ as in Definition 1.6. We prove Theorems 1.4 and 1.9 more less simultaneously making repeated use of Corollary 2.20.

4.1. Groups of low rank. It is immediate from [Z89, Cor. 7] that the reverse implication in Theorem 1.4 holds provided W has rank at most 3. Likewise Theorem 1.9 holds for W of rank at most 2.

4.2. Coxeter Groups. Let W be an irreducible Coxeter group. By [BC12], every Coxeter arrangement is inductively free. So once we have shown that $(\mathcal{A}(W)'', \kappa)$ is inductively free, so is $(\mathcal{A}(W), \delta)$, thanks to Corollary 2.20.

Lemma 4.1. *Let $\mathcal{A} = \mathcal{A}(A_{\ell-1})$ be the Coxeter arrangement of type $A_{\ell-1}$. Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Proof. For $\ell \leq 4$, the result follows from §4.1 and Corollary 2.21. So suppose that $\ell \geq 5$. Since the underlying Coxeter group is transitive on \mathcal{A} , there is no harm in fixing $H_0 = \ker(x_1 - x_2)$. Thanks to [OT92, Prop. 6.73], $\mathcal{A}'' \cong \mathcal{A}(A_{\ell-2})$. One readily checks that Ziegler's multiarrangement $(\mathcal{A}(A_{\ell-2}), \kappa)$ has defining polynomial

$$Q(\mathcal{A}(A_{\ell-2}), \kappa) = \prod_{2 \leq j \leq \ell-2} (x_1 - x_j)^2 \prod_{2 \leq i < j \leq \ell-2} (x_i - x_j).$$

It follows from [CR16, Cor. 1.2] that $(\mathcal{A}(A_{\ell-2}), \kappa)$ is inductively free. Since \mathcal{A} is inductively free, it follows from Corollary 2.20 that (\mathcal{A}, δ) is inductively free. \square

The following is the special case $r = 2$ in Proposition 3.11.

Lemma 4.2. *Let $\mathcal{A} = \mathcal{A}(B_\ell)$ be the Coxeter arrangement of type B_ℓ . Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Lemma 4.3. *Let $\mathcal{A} = \mathcal{A}(D_\ell)$ be the Coxeter arrangement of type D_ℓ for $\ell \geq 3$. Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Proof. We first prove that (\mathcal{A}'', κ) is inductively free. We argue by induction on $\ell \geq 3$. For $\ell = 3$, we have $\mathcal{A}(D_3) = \mathcal{A}(A_3)$ and so then the result follows from Lemma 4.1.

Now let $W = W(D_\ell)$ for $\ell \geq 4$ and suppose that the result holds for all instances of smaller rank. Since W is transitive on \mathcal{A} , we may fix $H_0 = \ker(x_1 - x_2)$ without loss. It follows from [OT92, Prop. 6.85] that $\mathcal{A}'' = \mathcal{A}_{\ell-1}^1(2)$.

One readily checks that Ziegler's multiarrangement $(\mathcal{A}(D_\ell)'', \kappa)$ has defining polynomial

$$Q(\mathcal{A}(D_\ell)'', \kappa) = Q(\mathcal{A}_{\ell-1}^1(2), \kappa) = x_1 \prod_{2 \leq j \leq \ell-1} (x_1^2 - x_j^2)^2 \prod_{3 \leq i < j \leq \ell-1} (x_i^2 - x_j^2).$$

According to Theorem 1.2, $(\mathcal{A}(D_\ell)'', \kappa)$ is free with

$$(4.4) \quad \exp(\mathcal{A}(D_\ell)'', \kappa) = \exp(\mathcal{A}_{\ell-1}^1(2), \kappa) = \{3, 5, \dots, 2(\ell-3) + 1, \ell-1\}.$$

By our induction hypothesis and Theorem 2.23, the sub-multiarrangement $(\mathcal{A}(D_{\ell-1})'', \kappa) \times \Phi_1$ of $(\mathcal{A}(D_\ell)'', \kappa)$ is inductively free. By (4.4), its set of exponents is

$$\exp((\mathcal{A}(D_{\ell-1})'', \kappa) \times \Phi_1) = \exp((\mathcal{A}_{\ell-2}^1(2), \kappa) \times \Phi_1) = \{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 0\}.$$

This forms the start of our induction table.

We first add and restrict to the hyperplanes $\ker(x_1 - x_{\ell-1})$ and $\ker(x_1 + x_{\ell-1})$ twice. Here we use Proposition 2.13 in order to determine the Euler multiplicity on the restrictions. We consider the special case when $\ker(x_1 + x_{\ell-1})$ is added and restricted to for the second time. Then in the notation of Proposition 2.13, when considering the localization $\mathcal{A}_X =$

$\{\ker(x_1 + x_{\ell-1}), \ker(x_1 - x_{\ell-1}), \ker x_1\}$, we have $|\mathcal{A}_X| = 3$, $|\mu_X| = 5$ and so Proposition 2.13(2) applies.

Note that each of the restrictions is isomorphic to $(\mathcal{A}_{\ell-2}^1(2), \kappa)$ which is inductively free, by our inductive hypothesis. Consequently, since in each case both (\mathcal{A}', μ') and (\mathcal{A}'', μ^*) are inductively free and $\exp(\mathcal{A}'', \mu^*) \subseteq \exp(\mathcal{A}', \mu')$, it follows that also (\mathcal{A}, μ) is inductively free.

$\exp(\mathcal{A}', \mu')$	α_H	$\exp(\mathcal{A}'', \mu^*)$
$\exp((\mathcal{A}_{\ell-2}^1(2), \kappa) \times \Phi_1)$ $= \{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 0\}$	$x_1 - x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
$\{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 1\}$	$x_1 + x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
$\{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 2\}$	$x_1 - x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
$\{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 3\}$	$x_1 + x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
$\{3, 5, \dots, 2(\ell-4) + 1, \ell-2, 4\}$	$x_2 - x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
\vdots	\vdots	\vdots
$\{3, 5, \dots, 2(\ell-4) + 1, 2(\ell-3) + 1, \ell-2\}$	$x_{\ell-2} + x_{\ell-1}$	$\exp(\mathcal{A}_{\ell-2}^1(2), \kappa)$
$\exp(\mathcal{A}(D_\ell)'', \kappa) = \{3, 5, \dots, 2(\ell-3) + 1, \ell-1\}$		

TABLE 1. Induction Table for $(\mathcal{A}(D_\ell)'', \kappa)$.

It follows from Table 1 that $(\mathcal{A}(D_\ell)'', \kappa)$ is inductively free. Since \mathcal{A} itself is inductively free, so is (\mathcal{A}, δ) , according to Corollary 2.20. \square

Lemma 4.5. *Let W be an irreducible Coxeter group of exceptional type of rank at least 3 and let $\mathcal{A} = \mathcal{A}(W)$ be the Coxeter arrangement of W . Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Proof. For W of type H_3 the result follows from Corollary 2.21.

For the remaining instances, we first prove that (\mathcal{A}'', κ) is inductively free. For W of type F_4 and H_4 , this follows from Corollary 3.14.

Since $W(E_6)$ and $W(E_7)$ are parabolic subgroups of $W(E_8)$, we obtain the result for the former from the latter, thanks to [OT92, Cor. 6.28], Lemma 2.16(ii), and Theorem 2.22.

The case for $W = W(E_8)$ is considerably more complicated. Since W is transitive on \mathcal{A} , we may take $H_0 = \ker x$. It follows from Proposition 3.10 that (\mathcal{A}'', ν) is free for every multiplicity ν with $1 \leq \nu \leq \kappa$. So we pick a filtration $\mathbb{1} = \nu_1 < \nu_2 < \dots < \nu_{28} < \nu_{29} = \kappa$ of free multiplicities. Since $\mathcal{B} := \mathcal{A}''$ is inductively free, by [BC12], it follows from Lemma 3.2 that (\mathcal{A}'', κ) is inductively free, once we have established that the corresponding multirestrictions (\mathcal{B}'', ν_i^*) are inductively free. Moreover, each multiplicity ν_i^* is just Ziegler's canonical multirestriction κ_1 stemming from the simple arrangement \mathcal{B} , by Corollary 3.8. Let $\mathcal{C} := \mathcal{B}''$. Since \mathcal{C} is inductively free, by [BC12], we can build an induction table by

starting with the simple arrangement \mathcal{C} and can increasing the multiplicities all the way to (\mathcal{C}, κ_1) . This is carried out in Table 2, where we indicate how to increase the multiplicities on the hyperplanes of the inductively free arrangement \mathcal{C} such that each resulting multiplicity is again a free multiplicity for \mathcal{C} . Note that in the first 16 steps of the table the Euler multiplicity μ^* of the multirestriction is just Ziegler's canonical restriction on the simple arrangement \mathcal{C} . The remaining two Euler multiplicities are small extensions of Ziegler's canonical multiplicity. In particular, here the order in which the hyperplanes are added plays a crucial role. There are precisely 3 different rank 5 multiarrangements that occur as restrictions in Table 2. We have checked that each of them is again inductively free, as required. However, we omit to include this data. We also do not give linear forms for α_H in the table, but give the number of the induction step at the beginning of each row.

Step	$\exp(\mathcal{A}', \mu')$	α_H	$\exp(\mathcal{A}'', \mu^*)$
0	$\exp(\mathcal{C}) = \{1, 7, 11, 13, 14, 17\}$...	$\{7, 11, 13, 14, 17\}$
	\vdots	\vdots	\vdots
15	$\{7, 11, 13, 14, 16, 17\}$...	$\{7, 11, 13, 14, 17\}$
16	$\{7, 11, 13, 14, 17, 17\}$...	$\{7, 11, 13, 17, 17\}$
	\vdots	\vdots	\vdots
20	$\{7, 11, 13, 17, 17, 18\}$...	$\{7, 11, 13, 17, 17\}$
21	$\{7, 11, 13, 17, 17, 19\}$...	$\{7, 11, 13, 17, 19\}$
	\vdots	\vdots	\vdots
26	$\{7, 11, 13, 17, 19, 22\}$...	$\{7, 11, 13, 17, 19\}$
	$\exp(\mathcal{C}, \kappa_1) = \{7, 11, 13, 17, 19, 23\}$		

TABLE 2. Induction Table for (\mathcal{C}, κ_1) .

We mention that the only types of restrictions which do occur throughout this induction for (\mathcal{A}'', κ) are (E_8, A_2) in rank 6, (E_8, A_3) in rank 5, (E_8, A_4) and (E_8, D_4) in rank 4 and $(E_8, A_1 \times D_5)$, (E_8, D_4) and (E_8, A_5) in rank 3, where we use the notation to indicate restrictions following [OS82, §3, App.] (cf. [OT92, §6.4, App. C]).

Finally, since (\mathcal{A}'', κ) is inductively free in each instance, it follows again from Corollary 2.20 that also (\mathcal{A}, δ) is inductively free. \square

4.3. The Monomial Groups $G(r, 1, \ell)$. For $W = G(r, 1, \ell)$ for $r, \ell \geq 2$, the result follows from Proposition 3.11.

4.4. Non-real inductively free reflection arrangements of exceptional type. Let W be irreducible of rank at least 3 of exceptional type so that $\mathcal{A} = \mathcal{A}(W)$ is inductively free. It then follows from Theorem 2.3 that W equals G_{25} , G_{26} , or G_{32} .

Lemma 4.6. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement where W is one of G_{25} , G_{26} , or G_{32} . Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are inductively free.*

Proof. Since G_{25} and G_{26} are both of rank 3, it follows from Corollary 2.21 that then $(\mathcal{A}(W), \delta)$ is inductively free. Since \mathcal{A}'' is of rank 2, also (\mathcal{A}'', κ) is inductively free.

For $W = G_{32}$ it follows from Corollary 3.14 that (\mathcal{A}'', κ) is inductively free. Owing to Corollary 2.20, also (\mathcal{A}, δ) is inductively free. \square

4.5. Non-inductively free reflection arrangements. Suppose that W is irreducible so that $\mathcal{A} = \mathcal{A}(W)$ is not inductively free. Then, according to Theorem 2.3, either $W = G(r, r, \ell)$, for $r, \ell \geq 3$ or W is of exceptional type G_{24} , G_{27} , G_{29} , G_{31} , G_{33} , or G_{34} .

The following is [HRS17, Prop. 2], its proof consists of an application of Theorem 2.22.

Lemma 4.7. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the monomial group $W = G(r, r, \ell)$ for $r \geq 3$ and $\ell \geq 5$. Then both (\mathcal{A}'', κ) and (\mathcal{A}, δ) are not inductively free.*

Lemma 4.8. *Let W be $G(r, r, 3)$, $G(r, r, 4)$, for $r \geq 3$ or of exceptional type G_{24} , G_{27} , G_{29} , G_{31} , G_{33} , or G_{34} . Then $(\mathcal{A}(W), \delta)$ is not inductively free.*

Proof. Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement for W as in the statement. We argue by induction on $m_0 \geq 1$. For $m_0 = 1$, the result follows from Theorem 2.3. So suppose that $m_0 > 1$ and that the claim is proved for smaller values for m_0 . Pick $H \in \mathcal{A}$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple of \mathcal{A} with respect to H . If $H = H_0$, then $(\mathcal{A}', \delta') = (\mathcal{A}, \delta_{H_0, m_0-1})$ which is not inductively free by inductive hypothesis.

Now suppose that $H \neq H_0$. In each case apart from G_{31} it follows from [HR15, Cor. 1.3, Cor. 2.4] that $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is not free for any choice of hyperplane. Therefore, in each of these instances, since \mathcal{A}' is not free, neither is (\mathcal{A}', δ') , by Theorem 1.7. In particular, (\mathcal{A}', δ') is not inductively free. Thus (\mathcal{A}', δ') is not inductively free for any choice of H and therefore, neither is (\mathcal{A}, δ) .

For W of exceptional type G_{31} , we argue as follows. Suppose that (\mathcal{A}, δ) is inductively free. Let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $(\mathcal{A}', \delta') = (\mathcal{A}', \delta|_{\mathcal{A}'})$. If (\mathcal{A}', δ') is free, then so is \mathcal{A}' , again thanks to Theorem 1.7. Continuing in this fashion removing further hyperplanes from the multiarrangement, we obtain a free chain of consecutive subarrangements of the simple arrangement $\mathcal{A}(G_{31})$ ending at the empty 4-arrangement. But this contradicts the fact that there is no such free chain of $\mathcal{A}(G_{31})$, according to the proof of [HR15, Lem. 3.5]. \square

Lemma 4.9. *Let W be $G(r, r, 3)$, $G(r, r, 4)$, for $r \geq 3$ or W is exceptional of type G_{29} or G_{31} . Then $(\mathcal{A}(W)'', \kappa)$ is inductively free.*

Proof. For $W = G(r, r, 3)$ the result is clear, thanks to Remark 2.18, and for $W = G_{31}$ the result follows from Corollary 3.14.

If $W = G(r, r, 4)$ or W is of type G_{29} , then $\mathcal{A}(W)''$ is inductively free, according to Theorem 2.4 (W is transitive on $\mathcal{A}(W)$ in each of these cases). Consequently, we may initialize an induction table in these instances with the simple inductively free arrangement $\mathcal{A}(W)''$ and add hyperplanes to reach the Ziegler multiplicity κ .

Let $W = G(r, r, 4)$. Then, by [OT92, Prop. 6.82], for any choice $H_0 \in \mathcal{A}$, we have $\mathcal{A}(W)'' \cong \mathcal{A}_3^1(r)$ which is inductively free, according to Theorem 2.4(ii). Fix $H_0 = \ker(x_1 - x_4)$. Thanks to [OT92, Cor. 6.86], we have

$$\exp \mathcal{A}(G(r, r, 4)) = \{1, r + 1, 2r + 1, 3(r - 1)\}$$

and therefore, by Theorem 1.2, $(\mathcal{A}(G(r, r, 4))'', \kappa)$ is free with

$$\exp(\mathcal{A}(G(r, r, 4))'', \kappa) = \{r + 1, 2r + 1, 3(r - 1)\}.$$

So in particular, $|\kappa| = 6r - 1$. Moreover, by [OT92, Prop. 6.85], we have

$$\exp \mathcal{A}(G(r, r, 4))'' = \exp \mathcal{A}_3^1(r) = \{1, r + 1, 2r - 1\}.$$

The defining polynomial for $(\mathcal{A}(G(r, r, 4))'', \kappa)$ is given by

$$Q(\mathcal{A}(G(r, r, 4))'', \kappa) = Q(\mathcal{A}_3^1(r), \kappa) = x_1^{r-1} (x_1^r - x_2^r)^2 (x_1^r - x_3^r)^2 (x_2^r - x_3^r).$$

Table 3 is the induction table for $(\mathcal{A}(G(r, r, 4))'', \kappa)$. Note that since $\mathcal{A}(G(r, r, 4))''$ is of rank 3, each of the restrictions in Table 3 is of rank 2 and so is inductively free. Let ζ be a primitive r^{th} root of unity.

There are two different types of steps in this induction table. The first one consists of adding a hyperplane $\ker(x_1 - \zeta^i x_2)$ for some i or $\ker(x_1 - \zeta^j x_3)$ for some j and the second one consists of adding $\ker x_1$.

In the first step, our defining polynomial is of the form

$$Q(\mathcal{A}, \mu) = x_1(x_2^r - x_3^r) \prod_{0 \leq i < r} (x_1 - \zeta^i x_2)^{k_i} (x_1 - \zeta^i x_3)^{l_i}$$

with $k_i, l_i \in \{1, 2\}$ for all i .

We claim that (\mathcal{A}, μ) is free with $\exp(\mathcal{A}, \mu) = \{1 + \sum_{0 \leq i < r} (k_i + l_i - 2), r + 1, 2r - 1\}$. This is true if $k_i = l_i = 1$ for all i , since then (\mathcal{A}, μ) is the simple arrangement $\mathcal{A}_3^1(r)$. For each hyperplane in this step the argument is essentially the same, so we only present it for $H = \ker(x_1 - \zeta^i x_2)$. Fix i and let $k_i = 1$. The defining polynomial of the restriction is

$$Q(\mathcal{A}^H, \mu^*) = x_1^r \prod_{0 \leq j < r} (x_1 - \zeta^j x_3)^2.$$

The Euler multiplicities can be computed using Proposition 2.13(2) and (3). A basis of $D(\mathcal{A}^H, \mu^*)$ is given by

$$\begin{aligned} \theta_1 &= r x_1^{r+1} D_1 + ((r + 1) x_1^r x_3 - x_3^{r+1}) D_3, \\ \theta_2 &= x_1^{r-1} (r x_1 x_3^{r-1} D_1 + (x_1^r + (r - 1) x_3^r) D_3). \end{aligned}$$

Therefore, we have $\exp(\mathcal{A}^H, \mu^*) = \{r + 1, 2r - 1\}$. The Addition Deletion Theorem 2.12 then yields that (\mathcal{A}, μ) is free with the desired exponents.

In the second step we consider defining polynomials of the form

$$Q(\mathcal{A}, \mu) = x_1^i (x_1^r - x_2^r)^2 (x_1^r - x_3^r)^2 (x_2^r - x_3^r),$$

for $1 \leq i \leq r - 2$.

We claim that (\mathcal{A}, μ) is free with $\exp(\mathcal{A}, \mu) = \{r+1, 2r+1, 2r-1+i\}$. We know from the first step that this is true for $i = 1$. Let $H = \ker x_1$. The restriction is given by

$$Q(\mathcal{A}^H, \mu^*) = x_2^{r+1} x_3^{r+1} (x_2^r - x_3^r).$$

Note that (\mathcal{A}^H, μ^*) coincides with $(\mathcal{A}(G(r, 1, 3))^{\ker x_1}, \kappa)$. Therefore, thanks to Theorem 1.2, (\mathcal{A}^H, μ^*) is free and $\exp(\mathcal{A}^H, \mu^*) = \{r+1, 2r+1\}$. Since the hypotheses of Theorem 2.12 are fulfilled, we get freeness of (\mathcal{A}, μ) along with the desired exponents for each i .

This completes the argument for $W = G(r, r, 4)$. We record the details of the induction in Table 3.

$\exp(\mathcal{A}', \mu')$	α_H	$\exp(\mathcal{A}'', \mu^*)$
$\exp \mathcal{A}_3^1(r) = \{1, r+1, 2r-1\}$	$x_1 - \zeta x_2$	$\{r+1, 2r-1\}$
$\{2, r+1, 2r-1\}$	$x_1 - \zeta^2 x_2$	$\{r+1, 2r-1\}$
\vdots	\vdots	\vdots
$\{r, r+1, 2r-1\}$	$x_1 - x_2$	$\{r+1, 2r-1\}$
$\{r+1, r+1, 2r-1\}$	$x_1 - \zeta x_3$	$\{r+1, 2r-1\}$
\vdots	\vdots	\vdots
$\{r+1, 2r, 2r-1\}$	$x_1 - x_3$	$\{r+1, 2r-1\}$
$\{r+1, 2r+1, 2r-1\}$	x_1	$\{r+1, 2r+1\}$
\vdots	\vdots	\vdots
$\{r+1, 2r+1, 2r-1+(r-3)\}$	x_1	$\{r+1, 2r+1\}$
$\exp(\mathcal{A}(G(r, r, 4))'', \kappa) = \{r+1, 2r+1, 3(r-1)\}$		

TABLE 3. Induction Table for $(\mathcal{A}(G(r, r, 4))'', \kappa)$.

Next let W be of type G_{29} . Since W is transitive on $\mathcal{A}(G_{29})$, we may take $H_0 = \ker x_4$. Then the defining polynomial for $(\mathcal{A}(G_{29})'', \kappa)$ is given by

$$\begin{aligned} Q(\mathcal{A}(G_{29})'', \kappa) &= (x_1 - x_2 - ix_3)^2 (x_1 - x_2)(x_2 - x_3)x_3^3 (x_1 - ix_2 - x_3)^2 (x_1 - x_2 + ix_3)^2 \\ &\quad (x_1 - x_3)(x_1 + ix_2 - ix_3)^2 x_2^3 (x_1 + ix_2 - x_3)^2 (x_1 - ix_2 + ix_3)^2 \\ &\quad x_1^3 (x_1 - ix_2 + x_3)^2 (x_1 - ix_2 - ix_3)^2 (x_1 + ix_2 + ix_3)^2 (x_1 + ix_2 + x_3)^2 \\ &\quad (x_1 + x_2 - ix_3)^2 (x_1 + x_2 + ix_3)^2 (x_2 + x_3)(x_1 + x_3)(x_1 + x_2). \end{aligned}$$

We present an induction table for $(\mathcal{A}(G_{29})'', \kappa)$ in Table 4. Note that since $\mathcal{A}(G_{29})''$ is of rank 3 each of the restrictions in Table 4 is of rank 2 and so is inductively free.

By the data in Tables 3, 4 and Remark 2.19, (\mathcal{A}'', κ) is inductively free in each instance. \square

Observe it follows from Theorem 2.3 and Lemmas 4.8 and 4.9 that the implication in Corollary 2.20 fails without the assumption that \mathcal{A}'' is inductively free.

$\exp(\mathcal{A}', \mu')$	α_H	$\exp(\mathcal{A}'', \mu^*), \mu^*$
$\exp \mathcal{A}(G_{29})'' = \{1, 9, 11\}$	$x_1 - x_2 - ix_3$	$\{9, 11\}, (1, 2, 2, 4, 2, 3, 4, 2)$
$\{2, 9, 11\}$	x_3	$\{9, 11\}, (3, 4, 3, 4, 3, 3)$
$\{3, 9, 11\}$	x_3	$\{9, 11\}, (3, 4, 3, 4, 3, 3)$
$\{4, 9, 11\}$	$x_1 - ix_2 - x_3$	$\{9, 11\}, (2, 1, 2, 2, 4, 3, 4, 2)$
$\{5, 9, 11\}$	$x_1 - x_2 + ix_3$	$\{9, 11\}, (1, 2, 2, 3, 2, 4, 4, 2)$
$\{6, 9, 11\}$	$x_1 + ix_2 - ix_3$	$\{9, 11\}, (2, 2, 3, 2, 1, 4, 2, 4)$
$\{7, 9, 11\}$	x_2	$\{9, 11\}, (3, 4, 3, 4, 3, 3)$
$\{8, 9, 11\}$	x_2	$\{9, 11\}, (3, 4, 3, 4, 3, 3)$
$\{9, 9, 11\}$	$x_1 + ix_2 - x_3$	$\{9, 11\}, (2, 2, 2, 3, 2, 1, 4, 4)$
$\{9, 10, 11\}$	$x_1 - ix_2 + ix_3$	$\{9, 11\}, (2, 2, 3, 2, 1, 4, 2, 4)$
$\{9, 11, 11\}$	x_1	$\{9, 11\}, (3, 3, 3, 3, 4, 4)$
$\{9, 11, 12\}$	x_1	$\{9, 11\}, (3, 3, 3, 3, 4, 4)$
$\{9, 11, 13\}$	$x_1 - ix_2 + x_3$	$\{9, 13\}, (2, 2, 2, 5, 1, 2, 3, 5)$
$\{9, 12, 13\}$	$x_1 - ix_2 - ix_3$	$\{9, 13\}, (2, 1, 2, 5, 3, 5, 2, 2)$
$\{9, 13, 13\}$	$x_1 + ix_2 + ix_3$	$\{9, 13\}, (2, 1, 2, 5, 3, 5, 2, 2)$
$\{9, 13, 14\}$	$x_1 + ix_2 + x_3$	$\{9, 13\}, (2, 5, 2, 5, 2, 3, 1, 2)$
$\{9, 13, 15\}$	$x_1 + x_2 - ix_3$	$\{9, 13\}, (2, 2, 3, 1, 2, 5, 5, 2)$
$\{9, 13, 16\}$	$x_1 + x_2 + ix_3$	$\{9, 13\}, (1, 2, 2, 5, 2, 3, 5, 2)$
$\exp(\mathcal{A}(G_{29})'', \kappa) = \{9, 13, 17\}$		

TABLE 4. Induction Table for $(\mathcal{A}(G_{29})'', \kappa)$.

Lemma 4.10. *Let W be of type G_{33} or G_{34} . Then $(\mathcal{A}(W)'', \kappa)$ is not inductively free.*

Proof. First consider the case of $W = G_{33}$. Let $\mathcal{A} = \mathcal{A}(W)$. Direct computations have shown that there is no chain of free multiplicities from the zero multiplicity on \mathcal{A}'' to κ . The maximal length of such chains is $22 < 44 = |\kappa|$. Hence (\mathcal{A}'', κ) is not inductively free.

Finally, let $W = G_{34}$ and $\mathcal{A} = \mathcal{A}(W)$. Then W admits a parabolic subgroup W_X of type $G(3, 3, 5)$, by [OT92, Table C.15]. Thanks to [OT92, Cor. 6.28], we have $\mathcal{A}(W_X) = \mathcal{A}(W)_X = \mathcal{A}_X$. Fix $H_0 \in \mathcal{A}_X$. Then considering the restriction with respect to H_0 , we have $(\mathcal{A}'')_X = (\mathcal{A}_X)''$. So we see that $((\mathcal{A}'')_X, \kappa_X) = ((\mathcal{A}_X)'', \kappa_X) = (\mathcal{A}(G(3, 3, 5))'', \kappa)$, by Lemma 2.16(ii). According to Lemma 4.7, the latter is not inductively free, thus neither is $(\mathcal{A}(G_{34})'', \kappa)$, owing to Theorem 2.22. \square

Finally, Theorems 1.4 and 1.9 follow from the results from §4.1 to §4.5 above.

Remark 4.11. The fact that $(\mathcal{A}(G_{33})'', \kappa)$ is not inductively free, proved in Lemma 4.10, and the data in the induction tables 2 and 4, were obtained by computational means.

Specifically, we retrieved relevant data about reflection groups from some GAP code provided by J. Michel [M15]; see also [S+97] and [GHL+96]. Subsequently, we made use of SAGE,

cf. [Ste15], and SINGULAR, cf. [GPS09], in order to calculate the various bases of modules of derivations involved.

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